

Coarse differentiation and quasi-isometries of a class of solvable Lie groups I

Irine Peng

Abstract

This is the first of two papers (the other one being [P]) which aim to understand quasi-isometries of a subclass of unimodular split solvable Lie groups. In the present paper, we show that locally (in a coarse sense), a quasi-isometry between two groups in this subclass is close to a map that respects their group structures.

Contents

1	Introduction	1
1.1	Proof outline	3
2	Preliminaries	4
2.1	Geometry of a certain class of solvable Lie groups	4
2.2	Notations	6
2.2.1	General remarks about paths, neighborhoods	6
2.2.2	Notations used in split abelian-by-abelian groups	7
3	Quasi-geodesics	10
3.1	Some facts about non-degenerate, split abelian-by-abelian groups	10
3.2	Efficient scale	12
3.3	Monotone scale	17
3.4	Occurrence of weakly monotone segments	26
3.5	Proof of Theorem 3.1	27
4	Inside of a box	30
4.1	Geometry of flats	31
4.2	Averaging	37
4.3	Proof of Theorem 1.1	42

1 Introduction

A (κ, C) *quasi-isometry* f between metric spaces X and Y is a map $f : X \rightarrow Y$ satisfying

$$\frac{1}{\kappa}d(p, q) - C \leq d(f(p), f(q)) \leq \kappa d(p, q) + C$$

with the additional property that there is a number D such that Y is the D neighborhood of $f(X)$. Two quasi-isometries f, g are considered to be equivalent if there is a number $E > 0$ such that $d(f(p), g(p)) \leq E$ for all $p \in X$.

From [A], any solvable Lie group \mathcal{L} has the form

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{L} \rightarrow \mathbb{R}^s \rightarrow 1$$

where \mathcal{U} largest connected normal nilpotent subgroup of \mathcal{L} , called its *nilradical*, and \mathbb{R}^s is the abelianization of its Cartan subgroup.

In a group G , an element $x \in G$ is called *exponentially distorted* if there are numbers c, ϵ such that for all $n \in \mathbb{Z}$,

$$\frac{1}{c} \log(|n| + 1) - \epsilon \leq \|x^n\|_G \leq c \log(|n| + 1) + \epsilon$$

where $\|x^n\|_G$ is the distance between the identity and x^n in G .

In the case of a connected, simply connected solvable Lie group G , Osin showed in [O] that the set of exponentially distorted elements forms a normal subgroup $R_{exp}(G)$ inside of the nilradical of G .

Motivated by the Gromov program of classifying groups up to quasi-isometries, we consider, in this two-part paper, quasi-isometries between connected, simply-connected unimodular solvable Lie group G whose exponential radical coincides with its nilradical and is a semidirect product between its abelian Cartan subgroup and its abelian nilradical that is 'irreducible' in some sense. (For example, is not a direct product with abelian factors). By applying the techniques introduced by Eskin-Fisher-Whyte in [EFW0], [EFW1], and [EFW2], we are able to show that

(Theorem ?? in [P](abridged)) Let G, G' be non-degenerate, unimodular, split abelian-by-abelian solvable Lie groups, and $\phi : G \rightarrow G'$ a κ, C quasi-isometry. Then ϕ is bounded distance from a composition of a left translation and a standard map.

Here a standard map is one that respect the factors in the semidirect product and their group structures. (See definition 2.1.1).

Consequently, we are able to see that
(Corollary ?? in [P])

$$\mathcal{QI}(G) = \left(\prod_{[\alpha]} \text{Bilip}(V_{[\alpha]}) \right) \rtimes \text{Sym}(G)$$

Here $V_{[\alpha]}$'s are subspaces of the nilradical, and $\text{Sym}(G)$ is a finite group, analogous to the Weyl group in reductive Lie groups. It reflects the symmetries of G . (See section 2.1)

Writing a non-degenerate, unimodular, split abelian-by-abelian solvable group as $G = \mathbf{H} \rtimes_{\varphi} \mathbf{A}$, where \mathbf{H} is the abelian nilradical and \mathbf{A} an abelian Cartan subgroup. We can also distinguish groups depending on whether the action of the Cartan subgroup on the nilradical (via φ) is diagonalizable or not.

(Corollary ?? in [P]) Let G, G' be non-degenerate, unimodular, split abelian-by-abelian solvable Lie groups where actions of their Cartan subgroups on the nilradicals are φ and φ' respectively. If φ is diagonalizable and φ' isn't, then there is no quasi-isometry between them.

When φ is diagonalizable, as an application the work by Dymarz [D] on quasi-conformal maps on the boundary of G , and a theorem of Mostow that says polycyclic groups are virtually lattices in a connected, simply connected solvable Lie group, we have

(Corollary ??, ?? in [P]) *In the case that φ is diagonalizable, if Γ is a finitely generated group quasi-isometric to $G = \mathbf{H} \rtimes_{\varphi} \mathbf{A}$, then Γ is virtually polycyclic, and is virtually a lattice in a unimodular semidirect product of \mathbf{H} and \mathbf{A} .*

Note that in the statement above we are not able to determine if the target semidirect product of \mathbf{H} and \mathbf{A} is actually G because the latter is a semidirect product of the same factors with some additional conditions, which we are not able to detect at this stage.

All the argument in this paper are local in nature and below is a description of the main result.

Let $G = \mathbf{H} \rtimes_{\varphi} \mathbf{A}$, $G' = \mathbf{H}' \rtimes_{\varphi'} \mathbf{A}'$ be connected, simply connected non-degenerate unimodular split solvable groups (See section 2.1 for definitions). We say a map from G to G' is *standard*, if it splits as a product map that respects φ and φ' (See definition 2.1.1).

A compact convex set $\Omega \subset \mathbb{R}^n$ determines a bounded set $\mathbf{B}(\Omega)$ in G (See section 2.2). Writing $\rho\Omega$ for the compact convex set obtained by scaling Ω by ρ from the barycenter of Ω , we show in this paper that

Theorem 1.1. *Let G, G' be non-degenerate, unimodular, split abelian-by-abelian Lie groups, and $\phi : G \rightarrow G'$ be a (κ, C) quasi-isometry. Given $0 < \delta, \eta < \tilde{\eta} < 1$, there exist numbers $L_0, m > 1, \varrho, \hat{\eta} < 1$ depending on $\delta, \eta, \tilde{\eta}$ and κ, C with the following properties:*

If $\Omega \subset \mathbf{A}$ is a product of intervals of equal size at least mL_0 , then a tiling of $\mathbf{B}(\Omega)$ by isometric copies of $\mathbf{B}(\varrho\Omega)$

$$\mathbf{B}(\Omega) = \bigsqcup_{i \in \mathbf{I}} \mathbf{B}(\omega_i) \sqcup \Upsilon$$

contains a subset \mathbf{I}_0 of \mathbf{I} with relative measure at least $1 - \nu$ such that

- (i) *For every $i \in \mathbf{I}_0$, there is a subset $\mathcal{P}^0(\omega_i)$ of $\mathbf{B}(\omega_i)$ of relative measure at least $1 - \nu'$*
- (ii) *The restriction $\phi|_{\mathcal{P}^0(\omega_i)}$ is within $\hat{\eta} \text{diam}(\mathbf{B}(\omega_i))$ Hausdorff neighborhood of a standard map $g_i \times f_i$.*

Here, ν, ν' and $\hat{\eta}$ all approach zero as $\tilde{\eta}, \delta$ go to zero. The measure of set Υ is at most δ' proportion of measure of $\mathbf{B}(\Omega)$, where δ' depends on δ and goes to zero as the latter approaches zero.

1.1 Proof outline

The idea of the proof is as follows. We employ the technique of ‘coarse differentiation’ to images of a particular family of geodesics (which fills up the set $\mathbf{B}(\Omega)$) in $\mathbf{B}(\Omega)$ to obtain the scale ρ on which those quasi-geodesics behave like certain simple geodesics. We are also able to obtain a tiling because the group G is unimodular and $\mathbf{B}(\Omega)$ have small boundary area compared to its volume. We then use the properties of the groups being non-degenerate, unimodular and split abelian-by-abelian to reach the conclusion on those smaller tiles.

Acknowledgement I would like to thank Alex Eskin for his patience and guidance. I also owe much to David Fisher for his help and support.

2 Preliminaries

In this section, we first describe the geometry of the subclass of unimodular solvable Lie group mentioned in Introduction, followed by a list of notations that will be used in the remaining of this paper.

2.1 Geometry of a certain class of solvable Lie groups

Non-degenerate, split abelian-by-abelian solvable Lie groups Let \mathfrak{g} be a (real) solvable Lie algebra, and \mathfrak{a} be a Cartan subalgebra. Then there are finitely many non-zero linear functionals $\alpha_i : \mathfrak{a} \rightarrow \mathbb{C}$ called *roots*, such that

$$\mathfrak{g} = \mathfrak{a} \oplus \bigoplus_{\alpha_i} \mathfrak{g}_{\alpha_i}$$

where $\mathfrak{g}_{\alpha_i} = \{x \in \mathfrak{g} : \forall t \in \mathfrak{a}, \exists n, \text{ such that } (ad(t) - \alpha_i(t)Id)^n(x) = 0\}$, Id is the identity map on \mathfrak{g} , and $ad : \mathfrak{g} \rightarrow Der_{\mathbb{R}}(\mathfrak{g})$ is the adjoint representation.

We say \mathfrak{g} is *split abelian-by-abelian* if \mathfrak{g} is a semidirect product of \mathfrak{a} and $\bigoplus_i \mathfrak{g}_{\alpha_i}$, and both are abelian Lie algebras; *unimodular* if the the roots sum up to zero; and *non-degenerate* if the roots span \mathfrak{a}^* . In particular, non-degenerate means that each α_i is real-valued, and the number of roots is at least the dimension of \mathfrak{a} . Being unimodular is the same as saying that for every $t \in \mathfrak{a}$, the trace of $ad(t)$ is zero. We extend these definitions to a Lie group if its Lie algebra has these properties.

Therefore a connected, simply connected solvable Lie group G that is non-degenerate, split abelian-by-abelian necessary takes the form $G = \mathbf{H} \rtimes_{\varphi} \mathbf{A}$ such that

- (i) both \mathbf{A} and \mathbf{H} are abelian Lie groups.
- (ii) the homomorphism $\varphi : \mathbf{A} \rightarrow Aut(\mathbf{H})$ is injective
- (iii) there are finitely many $\alpha_i \in \mathbf{A}^* \setminus 0$ which together span \mathbf{A}^* , and a decomposition of $\mathbf{H} = \bigoplus_i V_{\alpha_i}$
- (iv) there is a basis \mathcal{B} of \mathbf{H} whose intersection with each of V_{α_i} constitute a basis of V_{α_i} , such that for each $\mathbf{t} \in \mathbf{A}$, $\varphi(\mathbf{t})$ with respect to \mathcal{B} is a matrix consists of blocks, one for each V_{α_i} , of the form $e^{\alpha_i(\mathbf{t})} N(\alpha_i(t))$, where $N(\alpha_i(t))$ is an upper triangular with 1's on the diagonal and whose off-diagonal entries are polynomials of $\alpha_i(t)$. If in addition, G is unimodular, then $\varphi(\mathbf{t})$ has determinant 1 for all $\mathbf{t} \in \mathbf{A}$.

The *rank* of a non-degenerate, split abelian-by-abelian group G is defined to be the dimension of \mathbf{A} , and by a result of Cornulier [C], if two such groups are quasi-isometric, then they have the same rank.

Let Δ denotes the roots of G . For each $\alpha \in \Delta$, choose a basis $\{e_1^\alpha, e_2^\alpha, \dots, e_{n_\alpha}^\alpha\}$ in V_α such that $\varphi(\mathbf{t})|_{V_{\alpha_i}}$ is upper triangular for all $\mathbf{t} \in \mathbf{A}$. Also fix a basis $\{E_j\}$ in \mathbf{A} (for example, the duals of a subset of roots), and for each $\mathbf{t} \in \mathbf{A}$, write \mathbf{t}_j for its E_j coordinate. We coordinatize a point $(\sum_{\alpha \in \Delta} \sum_{j=1}^{n_\alpha} x_{j,\alpha} e_j^\alpha)(\mathbf{t}) \in \mathbf{H} \rtimes_{\varphi} \mathbf{A}$ by the $dim(G)$ -tuple of numbers $((\mathbf{x}_\alpha)_\alpha, (\mathbf{t}_j)) \in \mathbb{R}^{dim(G)}$, where $\mathbf{x}_\alpha = (x_{1,\alpha}, x_{2,\alpha}, \dots, x_{dim(V_\alpha),\alpha})$. In this coordinate system, a left invariant Riemannian metric at $((\mathbf{x}_\alpha)_\alpha, (\mathbf{t}_j)_j)$ is

$$\sum_j d(\mathbf{t}_j)^2 + \sum_{\alpha \in \Delta} e^{-2\alpha(\mathbf{t})} \sum_{i=1}^{\alpha_i} \left(dx_{i,\alpha} + \sum_{\iota=i+1}^{n_\alpha} P_{i,\iota}^\alpha(\alpha(-\mathbf{t})) dx_{\iota,\alpha} \right)^2$$

where $P_{i,\iota}^\alpha$ is a polynomial with no constant term. We see that the above Riemannian metric is bilipschitz to the following Finsler metric:

$$|d\mathbf{t}| + \sum_{\alpha \in \Delta} e^{-\alpha(\mathbf{t})} \sum_{i=1}^{n_\alpha} [1 + Q_{i,\alpha}(\alpha(-\mathbf{t}))] |dx_{i,\alpha}|$$

where $|d\mathbf{t}|$ means $\sum_j |d\mathbf{t}_j|$, and $Q_{i,\alpha}$ is sum of absolute values of polynomials with no constant terms.

Remark 2.1.1. *Since we defined our metric to be left-invariant, left multiplication by an element of G is an isometry. On the other hand, right multiplication typically distorts distance. For example, for points $p, q \in \mathbf{H}$, $\mathbf{t} \in \mathbf{A}$, $d(\mathbf{t}p, \mathbf{t}q) = d(p, q)$, but $d(p\mathbf{t}, q\mathbf{t})$ usually is some exponential-polynomial multiple of $d(p, q)$.*

Let $H_{s+1} = \mathbb{R}^s \rtimes_\psi \mathbb{R}$ be a non-unimodular solvable Lie group such that with respect to bases $\{e_i\}$, $\{E\}$ of \mathbb{R}^s and \mathbb{R} respectively, we have $\psi(tE) = e^{at}N(t)$, for all $t \in \mathbb{R}$. Here $a > 0$ and $N(t)$ is unipotent matrix (upper triangular with 1's on the diagonal) with polynomial entries. By giving a point $(\sum x_i e_i)(tE) \in H^{s+1}$ the coordinate of $(x_1, x_2, \dots, x_s, t)$, and argue as above we see that a left-invariant Finsler metric bilipschitz to a left-invariant Riemannian metric can be given as

$$|dt| + e^{-at} \sum_i [1 + P_i(at)] |dx_i| \quad (1)$$

where P_i is the sum of absolute values of polynomials with no constant terms.

The following consequence is immediate.

Lemma 2.1.1. *If G is non-degenerate, split abelian-by-abelian, then it can be QI embedded into $\prod_{\alpha \in \Delta} H_{\dim(V_\alpha)+1}$.*

Remark 2.1.2. *When $\psi(t)$ is diagonal, H_{s+1} is just the usual hyperbolic space.*

Proof. Because of the following relation

$$\begin{aligned} & \frac{1}{|\Delta|} \sum_{\alpha \in \Delta} \left(|d\alpha(\mathbf{t})| + e^{-\alpha(\mathbf{t})} \sum_{i=1}^{n_\alpha} [1 + Q_{i,\alpha}(\alpha(-\mathbf{t}))] |dx_{i,\alpha}| \right) \leq |d\mathbf{t}| + \sum_{\alpha \in \Delta} e^{-\alpha(\mathbf{t})} \sum_{i=1}^{n_\alpha} [1 + Q_{i,\alpha}(\alpha(-\mathbf{t}))] |dx_{i,\alpha}| \\ & \leq \sum_{\alpha \in \Delta} \left(|d\alpha(\mathbf{t})| + e^{-\alpha(\mathbf{t})} \sum_{i=1}^{n_\alpha} [1 + Q_{i,\alpha}(\alpha(-\mathbf{t}))] |dx_{i,\alpha}| \right) \end{aligned}$$

□

To understand the geometry of H_{s+1} better, we can assume without loss of generality that $a = 1$, and note that the Finsler metric in equation (1) is quasi-isometric to one given by $dt + e^{-t}Q(t)d\mathbf{x}$ for some polynomial $Q(t)$. Since exponential grows faster than polynomials, for any large positive

number x , there is a t_0 such that $e^{-t}Q(t)x \leq 1$ for all $t \geq t_0$, and we see that a function q.i. to the metric on H_{s+1} is the following

$$d((\mathbf{x}_1, t_1), (\mathbf{x}_2, t_2)) = \begin{cases} |t_1 - t_2| & \text{if } e^{-t_i}Q(t_i)|\mathbf{x}_1 - \mathbf{x}_2| \leq 1 \text{ for some } i = 1, 2; \\ U_Q(|\mathbf{x}_1 - \mathbf{x}_2|) - (t_1 + t_2) & \text{otherwise} \end{cases} \quad (2)$$

where $U_Q(|\mathbf{x}_1 - \mathbf{x}_2|) = t_0$ satisfies

$$e^{-t_0}Q(t_0)|\mathbf{x}_1 - \mathbf{x}_2| = 1$$

Furthermore, the following relation

$$\frac{1}{e^t} < \frac{Q(t)}{e^t} < \frac{e^{1/2t}}{e^t} \quad \text{for } t \text{ sufficiently large}$$

and the fact that both e^{-t} and $Q(t)e^{-t}$ are decreasing functions when t becomes big enough means that we have the following inequalities for their inverses:

$$\ln(x) - C_Q \leq U_Q(x) \leq 2\ln(x) + C_Q \quad \text{for } x > 1 \quad (3)$$

for some constant C depends only on the polynomial Q .

Back to the description of G , we declare two roots equivalent if they are positive multiples of each other, and write $[\Xi]$ for the equivalence class containing $\Xi \in \Delta$. A left translate of $V_{[\Xi]} = \oplus_{\sigma \in [\Xi]} V_\sigma$ will be called a *horocycle of root class* $[\Xi]$.

A left translate of \mathbf{H} , or a subset of it, is called a *flat*. For two points $p, q \in \mathbf{H}$ with coordinates $(\mathbf{x}_\alpha)_{\alpha \in \Delta}$ and $(\mathbf{y}_\alpha)_{\alpha \in \Delta}$, we compute subsets of $p\mathbf{H}$ and $q\mathbf{H}$ that are within distance 1 of each other according to the embedded metric in Lemma 2.1.1, as the p and q translate of the subset of \mathbf{A} :

$$\bigcap_{\alpha \in \Delta: \ln(|\mathbf{x}_\alpha - \mathbf{y}_\alpha|) \geq 1} \alpha^{-1}[U_\alpha(|\mathbf{x}_\alpha - \mathbf{y}_\alpha|), \infty]$$

As the roots sum up to zero in a non-degenerate, unimodular, split abelian-by-abelian group, the set where two flats come together can be empty, i.e. the two flats have no intersection. If it is not empty, then the equation above says that it is an unbounded convex subset of \mathbf{A} bounded by hyperplanes parallel to root kernels.

Definition 2.1.1. *Let G, G' be non-degenerate, split abelian-by-abelian Lie groups. A map from G to G' or a subset of them, is called *standard* if it takes the form $f \times g$, where $g : \mathbf{H} \rightarrow \mathbf{H}'$ sends foliation by root class horocycles of G to that of G' , and $f : \mathbf{A} \rightarrow \mathbf{A}$ sends foliations by root kernels of G to that of G' .*

Remark 2.1.3. *Note that when G has at least $\text{rank}(G) + 1$ many root kernels, the condition on f means that f is affine, and when G is rank 1, the condition on f is empty.*

2.2 Notations

2.2.1 General remarks about paths, neighborhoods

Division of a curve The word 'scale' shall mean a number $\rho \in (0, 1]$. We will often examine a quasi-geodesic on different 'scales', and see if the quasi-geodesic 'on that scale' satisfies certain

properties. This roughly means that we subdivide the quasi-geodesic into subsegments whose lengths are ρ times the length of the original one, and see if each one of them satisfies certain properties.

In practice, however, instead of dealing with 'length', we use 'distance between end points' of a curve. More precisely, let $\zeta : [a, b] \rightarrow Y$ be rectifiable curve.

- Given $r > 0$, we can divide ζ into subsegments whose end points are r apart.
More precisely, $\hat{\mathcal{S}}(\zeta, r) = \{q_i\}_{i=1}^{n_r}$, is the set of the dividing points on ζ , where $q_0 = \zeta(a)$, $q_{n_r} = \zeta(b)$, and

$$\zeta^{-1}(q_{i+1}) = \min\{t \geq \zeta^{-1}(q_i) \mid d(\zeta(t), q_i) = r\}$$

- Given two points $p, q \in \zeta$, we write $\zeta_{[p, q]}$ for the part of ζ between p and q . Define $\mathcal{S}(\zeta, r) = \{\zeta_{[q_i, q_{i+1}]}\}$, to be the set of subsegments after division.
- Let \mathbf{P} be a statement. Define $\mathcal{S}(\zeta, r, \mathbf{P}) = \{\zeta^i \in \mathcal{S}(\zeta, r) \mid \zeta^i \text{ satisfies } \mathbf{P}\}$ to be those subsegments satisfying statement \mathbf{P} .
- We write $|\zeta|$ for the distance between end points of ζ , and $\|\zeta\|$ denotes for the length of ζ .

Neighborhoods of a set We write $B(p, r)$ for the ball centered at p of radius r , and $N_c(A)$ for the c neighborhood of the set A . We also write $d_H(A, B)$ for the Hausdorff distance between two sets A and B . If $\Omega \subset \mathbb{R}^k$ is a bounded compact set, and $r \in \mathbb{R}$, we write $r\Omega$ for the bounded compact set that is scaled from Ω with respect to the barycenter of Ω .

Given a set X , a point $x_0 \in X$, the (η, C) linear neighborhood of X with respect to x_0 is the set $\{y, s.t. \exists \hat{x} \in X, d(y, \hat{x}) = d(y, X) \leq \eta d(\hat{x}, x_0) + C\}$. Equivalently it is the set $\bigcup_{x \in X} B(x, \eta d(x, x_0) + C)$. By (η, C) linear neighborhood of a set X , we mean the (η, C) linear neighborhood of X with respect to some $x_0 \in X$.

If a quasi-geodesic λ is within (η, C) linear (or just η -linear) neighborhood of a geodesic segment γ , where $\eta \ll 1$ and $C \ll \eta|\lambda|$, then we say that λ admits a geodesic approximation by γ .

2.2.2 Notations used in split abelian-by-abelian groups

Let $G = \mathbf{H} \rtimes \mathbf{A}$ stands for a non-degenerate, split abelian-by-abelian group, and Δ denotes for its roots. Fix a point $p \in G$. We define the following:

- For $\alpha \in \Delta$ a root, we write \vec{v}_α for the dual of α_i of norm 1 with respect to the usual Euclidean metric. (This is really a function on root classes.)
- Given $\vec{v} \in \mathbf{A}$, we define

$$\begin{aligned} W_{\vec{v}}^+ &= \oplus_{\Xi(\vec{v}) > 0} V_\Xi \\ W_{\vec{v}}^- &= \oplus_{\Xi(\vec{v}) < 0} V_\Xi \\ W_{\vec{v}}^0 &= \oplus_{\Xi(\vec{v}) = 0} V_\Xi \end{aligned}$$

- Let $\ell \in \mathbf{A}^*$, we define $W_\ell^+, W_\ell^-, W_\ell^0$, as $W_{\vec{v}_\ell}^+, W_{\vec{v}_\ell}^-, W_{\vec{v}_\ell}^0$ respectively, where $\vec{v}_\ell \in \mathbf{A}$ is the dual of ℓ .
- By the walls based at p , we mean the set $p \bigcup_{\Xi} \ker(\Xi)$.

- By a *geodesic segment through p* , we mean a set $p\overline{AB}$, where \overline{AB} is a directed line segment in \mathbf{A} . By direction of a directed line segment in Euclidean space, we mean a unit vector with respect to the usual Euclidean metric, and by direction of $p\overline{AB}$ we mean the direction of \overline{AB} .
- For $i = 2, 3, \dots, n-1$, by *i -hyperplane through p* , we mean a set pS , where $S \subset \mathbf{A}$ is an i -dimensional linear subspace or an intersection between an i -dimensional linear subspace with a convex set.
- Let $\pi_A : G = \mathbf{H} \rtimes \mathbf{A} \longrightarrow \mathbf{A}$ be the projection onto the \mathbf{A} factor as $(\mathbf{x}, \mathbf{t}) \mapsto \mathbf{t}$
- For each root α_i , define $\pi_{\alpha_i} : G \longrightarrow V_{\alpha_i} \rtimes \langle \vec{v}_{\alpha_i} \rangle$ as $(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{|\Delta|})\mathbf{t} \mapsto (\mathbf{x}_i, \alpha_i(\mathbf{t})\vec{v}_{\alpha_i})$. We refer to negatively curved spaces $V_{\alpha_i} \rtimes \langle \vec{v}_{\alpha_i} \rangle$ or $V_{[\alpha]} \rtimes \langle \vec{v}_{\alpha} \rangle$ as weight (or root) hyperbolic spaces.

Now assume in addition that G is unimodular. Fix a net \mathbf{n} in G . For $\alpha \in \Delta$, let $b(r) \subset V_{\alpha}$ be maximal product of intervals of size r , $[0, r]^{\dim(V_{\alpha})}$, and since \mathbf{H} is the direct sum of those V_{α} 's, we write $\prod_{\alpha \in \Delta} b(r_{\alpha})$ for the product of those $b(r_{\alpha})$'s as α ranging over all roots. In other words, $\prod_{\alpha \in \Delta} b(r_{\alpha})$ is just product of intervals in \mathbf{H} where interval length is r_{α} in V_{α} .

Let $\Omega \subset \mathbf{A}$ be a convex compact set with non-empty interior, e.g. a product of intervals or a convex polyhedra. Without loss of generality assume its barycenter is the identity of \mathbf{A} . We define the *box associated to Ω* , $\mathbf{B}(\Omega)$, as the set $\left(\prod_{j=1}^{|\Delta|} b(e^{\max(\alpha_j(\Omega))}) \right) \Omega$.

Remark 2.2.1. A box $\mathbf{B}(\Omega)$ as defined above is just a union of left translates of $\Omega \subset \mathbf{A}$ by a subset of \mathbf{H} (product of intervals) whose size is determined by Ω . The size of the intervals were chosen so that a large proportion of points in the box $\mathbf{B}(\Omega)$ lie on a quadrilateral (see Definition 4.1.2). In the definition above we have defined this subset of \mathbf{H} as a product of intervals, but this is just a choice of convenience so that it is simple to describe the size of this subset in \mathbf{H} in terms of Ω .

Associate to the box $\mathbf{B}(\Omega)$, we use the following notations:

- $\mathcal{L}(\Omega)[m]$ (or $\mathcal{L}(\mathbf{B}(\Omega))[m]$) for the set of geodesics in $\mathbf{B}(\Omega)$ whose π_A images begin and end at points of $\partial\Omega$ such that the ratio between its length and the diameter of Ω lies in the interval $[1/m, m]$.
- For $i = 2, 3, \dots, n$, write $\mathcal{L}_i(\Omega)[m_i]$ (or $\mathcal{L}_i(\mathbf{B}(\Omega))[m_i]$) for the set of i dimensional hyperplanes in $\mathbf{B}(\Omega)$ such that the ratio between its diameter and the diameter of Ω lies in the interval $[1/m_i, m_i]$.
- $\mathcal{P}(\Omega)$ (or $\mathcal{P}(\mathbf{B}(\Omega))$) for the set of points in $\mathbf{B}(\Omega)$.
- Let S be an element of $\bigcup_{i=2}^n \mathcal{L}_i(\Omega) \cup \mathcal{L}(\Omega) \cup \mathcal{P}(\Omega)$. We write $L(S)$, $L_i(S)$ for subset of $\mathcal{L}(\Omega)$, $\mathcal{L}_i(\Omega)$ contained or containing S , and $P(S)$ for the subset of $\mathcal{P}(\Omega)$ contained in S .

Remark 2.2.2. As we are interested in a given quasi-isometry $\phi : G \rightarrow G'$ which implicitly implies particular choices of nets $n \subset G$, $n' \subset G'$, we will primarily consider ϕ as a map from n to n'^{\dagger} . Let $\hat{p} : G \rightarrow n$ that assigns $x \in G$, a closest net point. In this way we tend to think of a set $K \subset G$ not so much as a subset of the Lie group G , but as a subset of n via the identification of K and $\hat{p}(K)$.

In particular, the set of hyperplanes and points associated to a box as defined above would be considered finite sets for us.

[†]But then any two nets are bounded distance apart, and a bounded modification does not change the quasi-isometry class of ϕ , so whatever argument we make for n and n' are valid for other choices of nets as well.

We now use boxes to produce a sequence of Følner sets.

Lemma 2.2.1. *Let $G = \mathbf{H} \rtimes \mathbf{A}$ be a non-degenerate, unimodular, split abelian-by-abelian Lie group. Let $\Omega \subset \mathbf{A}$ be compact convex with non-empty interior. Then, $\mathbf{B}(r\Omega)$, $r \rightarrow \infty$ is a Følner sequence. The volume ratio between $N_\epsilon(\partial(\mathbf{B}(r\Omega)))$ and $\mathbf{B}(r\Omega)$ is $O(\epsilon/\text{diam}(\mathbf{B}(r\Omega)))^\dagger$.*

Proof. For each root α_j , write $\alpha_j(\Omega) = [b_j, a_j]$. Since the sum of roots is zero, the volume element is $\wedge_j d\mathbf{x}_j \wedge dt$. Therefore $\text{vol}(\mathbf{B}(r\Omega)) = \left(\prod_j e^{ra_j}\right) r^n |\Omega|$. On the other hand, the area of the boundary is

$$\left| \partial \left(\prod_j [0, e^{ra_j}] (r\Omega) \right) \right| = \underbrace{\left| \partial \left(\prod_j [0, e^{ra_j}] \right) (r\Omega) \right|}_{(1)} + \underbrace{\left| \left(\prod_j [0, e^{ra_j}] \right) \partial (r\Omega) \right|}_{(2)}$$

We estimate the size of each term:

$$\begin{aligned} (2) : \left| \left(\prod_j [0, e^{ra_j}] \right) \partial (r\Omega) \right| &= \left(\prod_j e^{ra_j} \right) r^{n-1} |\partial\Omega| \\ (1) : \left| \partial \left(\prod_j [0, e^{ra_j}] \right) (r\Omega) \right| &= 2 \sum_j \int_{\mathbf{t} \in r\Omega} \underbrace{\int_{\mathbf{x}_1 \in b(e^{ra_1}), \dots, \mathbf{x}_i \in b(e^{ra_i})} e^{-\alpha_1(\mathbf{t})} d\mathbf{x}_1 \dots e^{-\alpha_i(\mathbf{t})} d\mathbf{x}_i \dots dt}_{i \neq j} \\ &= 2 \sum_j \left(\prod_{i \neq j} e^{ra_i} \int_{\mathbf{t} \in r\Omega} e^{\alpha_j(\mathbf{t})} d\mathbf{t} \right) \\ &\leq 2 \sum_j \left(\prod_{i \neq j} e^{ra_i} (e^{ra_j} - e^{rb_j}) |Proj_{ker(\alpha_j)}(r\Omega)| \right) \\ &= 2 \left(\prod_i e^{ra_i} \right) r^{n-1} \left(\sum_j |Proj_{ker(\alpha_j)}(\Omega)| (1 - e^{-(ra_j - rb_j)}) \right) \\ &\leq 2|\Delta| \left(\prod_i e^{ra_i} \right) r^{n-1} \max_j |Proj_{ker(\alpha_j)}(\Omega)| \end{aligned}$$

□

Remark 2.2.3. *The same calculation as above shows that for any set \tilde{B} of the form $\Lambda \rtimes \Omega$, where $\Lambda \subset \mathbf{H}$, $\Omega \subset \mathbf{A}$, the ratio of volumes of $N_\epsilon(\partial\tilde{B})$ and that of \tilde{B} is $O(\epsilon/\text{diam}(\tilde{B}))$.*

[†]because the ratio of volumes of $\partial\Omega$ to Ω is roughly $\frac{1}{\text{diam}(\Omega)}$, and $r\text{diam}(\Omega) = \text{diam}(r\Omega)$

3 Quasi-geodesics

The purpose of this section is to prove

Theorem 3.1. *Let G, G' be non-degenerate, unimodular, split abelian-by-abelian Lie groups, and $\phi : G \rightarrow G'$ a (κ, C) quasi-isometry. Given $0 < \delta, \eta < \tilde{\eta} < 1$, there are numbers $L_0, m > 1$ and $0 < \rho < 1$ depending on δ, η, κ, C with the following properties:*

If $\Omega \subset \mathbf{A}$ is a product of intervals of equal size at least mL_0 , then a tiling of $\mathbf{B}(\Omega)$ by isometric copies of $\mathbf{B}(\rho\Omega)$

$$\mathbf{B}(\Omega) = \bigsqcup_{j \in \mathbf{J}} \mathbf{B}(\Omega_j) \sqcup \Upsilon$$

contains a subset \mathbf{J}_0 whose measure is at least $1 - \vartheta$ times that of \mathbf{J} such that:

- (i) *For all $j \in \mathbf{J}_0$, there is a subset $\mathcal{L}_j^0 \subset \mathcal{L}(\Omega_j)[m]$, whose measure is at least $1 - \varkappa$ times that of $\mathcal{L}(\Omega_j)$*
- (ii) *If $\zeta \in \mathcal{L}_j^0$, then $\phi(\zeta)$ is within η -linear neighborhood of a geodesic segment which makes an angle at least $\sin^{-1}(\tilde{\eta})$ with root kernels.*

Here ϑ, \varkappa approach zero as $\tilde{\eta} \rightarrow 0$. The measure of set Υ is at most δ' proportion of measure of $\mathbf{B}(\Omega)$, where δ' depends on δ and goes to zero as the latter approaches zero.

3.1 Some facts about non-degenerate, split abelian-by-abelian groups

In this subsection, G denotes a non-degenerate, split abelian-by-abelian group. By Lemma 2.1.1, we can use the embedded metric on G . We will use the metric property of those H_{s+1} spaces to obtain the following proposition, which basically says that if a quasi-geodesic in G is long, then its projection in \mathbf{A} has to be long as well.

Proposition 3.1.1. *Let $\zeta : [0, L] \rightarrow G$ be a (κ, C) quasi-geodesic segment. Suppose $\{\pi_A(\zeta(t))\}$ lies in a ball of diameter s . Then for any $p, q \in \zeta$, $d(p, q) \leq \hbar s$, where \hbar is a constant that depends only on the number of roots.*

Corollary 3.1.1. *(assumptions as in Proposition 3.1.1) If there are two points p, q on ζ such that $d(p, q) > \hbar s$, then there must be a point $r \in [\zeta^{-1}(p), \zeta^{-1}(q)]$ such that $d(\pi_A(p), \pi_A(\zeta(r))) > s$.*

To prove Proposition 3.1.1, we need the following two lemmas whose verifications can be found in the Appendix.

In $H_{n'+1} = \mathbb{R}^{n'} \rtimes \mathbb{R}$, we write h for the projection onto the \mathbb{R} factor.

Lemma 3.1.1. *Let $\eta : [a, b] \rightarrow H_{n'+1}$ be a continuous path such that*

- *The image of $h \circ \eta$ is contained in an interval of length no bigger than s , where $s > \kappa(C_{H_{n'+1}})^2 (> 2)$. Here $C_{H_{n'+1}}$ is a constant depending only on $H_{n'+1}$ (as in equation (3)).*
- *whenever $i_1 \leq i_2 \leq \dots i_n \in [a, b]$,*

$$\frac{\sum_j d(\eta(i_j), \eta(i_{j+1}))}{d(\eta(i_1), \eta(i_n))} \leq 2\kappa$$

Then, for any two points $p, q \in \eta([a, b])$, $d(p, q) \leq \hat{C}(2\kappa)s$, where \hat{C} depends only on $C_{H_{n'}+1}$.

Proof. see Appendix □

Lemma 3.1.2. Let $a, b \geq 0$, $A, B > 0$. Suppose $\frac{a+b}{A+B} = c_\alpha \frac{a}{A} + c_\beta \frac{b}{B}$, with $c_\alpha + c_\beta = 1$. Suppose $c_\alpha \geq c_\beta$, then $A \geq B$.

Proof. see Appendix □

Proof. of Proposition 3.1.1

We proceed by induction on the number of roots. The base step where there is just one root is Lemma 3.1.1. Since ζ is a (κ, c) quasi-geodesic, for any $i_0 \leq i_1 \leq i_2 \leq i_3 \dots i_n \in [0, L]$, we must have

$$\frac{\sum_j d(\zeta(i_j), \zeta(i_{j+1}))}{d(\zeta(i_0), \zeta(i_n))} \leq 2\kappa \quad (4)$$

We recall from Lemma 2.1.1 that $d(\cdot, \cdot) = \sum_{l=1}^{|\Delta|} d^{\alpha_l}(\pi_{\alpha_l}(\cdot), \pi_{\alpha_l}(\cdot))$, and proceed to simplify equation (4) by writing $d^{\alpha_l}(\pi_{\alpha_l}(\zeta(i_j)), \pi_{\alpha_l}(\zeta(i_{j+1})))$ as $d_j^{\alpha_l}$, and $d^{\alpha_l}(\pi_{\alpha_l}(\zeta(i_0)), \pi_{\alpha_l}(\zeta(i_n)))$ as d^{α_l} .

Now equation (4) becomes

$$\frac{\sum_j (d_j^{\alpha_1} + d_j^{\alpha_2} + \dots + d_j^{\alpha_{|\Delta|}})}{d^{\alpha_1} + d^{\alpha_2} + \dots + d^{\alpha_{|\Delta|}}} \leq 2\kappa$$

- Suppose for some weight, let's say α_1 , we have

$$\frac{\sum_j d_j^{\alpha_1} + \sum_j (d_j^{\alpha_2} + d_j^{\alpha_3} \dots + d_j^{\alpha_{|\Delta|}})}{d^{\alpha_1} + (d^{\alpha_2} + d^{\alpha_3} + \dots + d^{\alpha_{|\Delta|}})} = c_\alpha \frac{\sum_j d_j^{\alpha_1}}{d^{\alpha_1}} + c_\beta \frac{\sum_j (d_j^{\alpha_2} + d_j^{\alpha_3} \dots + d_j^{\alpha_{|\Delta|}})}{(d^{\alpha_2} + d^{\alpha_3} + \dots + d^{\alpha_{|\Delta|}})} \quad (5)$$

with $c_\alpha + c_\beta = 1$, and $c_\alpha \geq c_\beta$. Therefore

- ★ $c_\alpha \geq 1/2$. Since equation (5) is bounded above by 2κ , we now have an upper bound for the first term:

$$\frac{1}{2} \frac{\sum_j d_j^{\alpha_1}}{d^{\alpha_1}} \leq 2\kappa$$

That is, $\{\pi_\alpha(\zeta(i_j))\}$ are points whose heights in the α weight hyperbolic space lie in an interval of width no bigger than s (because $\pi_A(\zeta(i_j))$ lies in a ball of diameter s), and

$$\frac{\sum_j d^{\alpha_1}(\pi_{\alpha_1}(\zeta(i_j)), \pi_{\alpha_1}(\zeta(i_{j+1})))}{d^{\alpha_1}(\pi_\alpha(\zeta(i_0)), \pi_\alpha(\zeta(i_n)))} \leq 4\kappa$$

By Lemma 3.1.1, $d^\alpha(\pi_\alpha(\zeta(i_0)), \pi_\alpha(\zeta(i_n))) \leq \hat{C}(4\kappa)s$

- ★ Since $c_\alpha \geq c_\beta$, Lemma 3.1.2 says $d^{\alpha_1} \geq \sum_{l=2}^{|\Delta|} d^{\alpha_l}$, which makes $d(\zeta(i_0), \zeta(i_n)) = d^{\alpha_1} + \sum_{l=2}^{|\Delta|} d^{\alpha_l} \leq 2\hat{C}(4\kappa)s = 2^2 \hat{C}(2\kappa)s$

- If the first possibility doesn't occur, then for every weight $\alpha_{i'}$, we must have

$$\frac{\sum_j (d_j^{\alpha_1} + d_j^{\alpha_2} \cdots d_j^{\alpha_{|\Delta|}})}{d^{\alpha_1} + d^{\alpha_2} + \cdots d^{\alpha_{|\Delta|}}} = c_{\alpha_{i'}} \frac{\sum_j d_j^{\alpha_{i'}}}{d^{\alpha_{i'}}} + c_{\beta_{i'}} \frac{\sum_j (d_j^{\alpha_1} + d_j^{\alpha_2} \cdots d_j^{\alpha_{i'-1}} + d_j^{\alpha_{i'+1}} \cdots d_j^{\alpha_{|\Delta|}})}{\sum_{l \neq i'} d^{\alpha_l}} \quad (6)$$

with $c_{\alpha_{i'}}, c_{\beta_{i'}} \geq 0$, $c_{\alpha_{i'}} + c_{\beta_{i'}} = 1$, BUT $c_{\alpha_{i'}} \leq c_{\beta_{i'}}$. We fix such an i' . Then

- ★ $c_{\beta_{i'}} \geq 1/2$. Since the equation (6) is bounded above by 2κ , we obtain an upper bound for the second term on the right hand side:

$$\frac{\sum_j (d_j^{\alpha_1} + d_j^{\alpha_2} \cdots d_j^{\alpha_{i'-1}} + d_j^{\alpha_{i'+1}} \cdots d_j^{\alpha_{|\Delta|}})}{\sum_{l \neq i'} d^{\alpha_l}} \leq 4\kappa$$

By inductive hypothesis,

$$\sum_{l \neq i'} d^{\alpha_l} (\pi_{\alpha_l}(\zeta(i_0)), \pi_{\alpha_l}(\zeta(i_n))) = \sum_{l \neq i'} d^{\alpha_l} \leq 2^{2(|\Delta|-2)} \hat{C}(4\kappa)s$$

- ★ Finally, since $c_{\alpha_{i'}} \leq c_{\beta_{i'}}$, Lemma 3.1.2 says $d^{\alpha_{i'}} \leq \sum_{l \neq i'} d^{\alpha_l}$ which means $d(\zeta(i_0), \zeta(i_n)) = d^{\alpha_{i'}} + \sum_{l \neq i'} d^{\alpha_l} \leq 2^{2(|\Delta|-2)} \hat{C}(4\kappa)s = 2^{2(|\Delta|-1)} \hat{C}(2\kappa)s$

□

3.2 Efficient scale

This subsection is based on definition 4.5 and lemma 4.6 in [EFW0], where ϵ -efficiency was defined. Here we note the consequence of an efficient segment in a non-degenerate, split abelian-by-abelian group.

Definition 3.2.1. (ϵ -efficient at scale \tilde{r}) *Let Y be a metric space, and $\lambda : [0, L] \rightarrow Y$ a rectifiable curve. We say that λ is ϵ -efficient at scale \tilde{r} , $0 < \tilde{r} \leq 1$ if*

$$\sum_j d(p_j, p_{j+1}) \leq (1 + \epsilon)d(\lambda(0), \lambda(L)), \text{ where } \{p_j\} = \hat{S}(\lambda, \tilde{r}d(\lambda(0), \lambda(L)))$$

Remark 3.2.1. *Note that being efficient at scale r does necessarily not imply efficient at all sales $\tilde{r} < r$.*

Efficiency provides with us the closest description of being ‘straight’ in \mathbb{R}^n , whose meaning is made precise by the following lemma.

Lemma 3.2.1. *If $\lambda : [a, b] \rightarrow \mathbb{R}^n$ is ϵ -efficient at scale r , then $d_H(\lambda, \overline{\lambda(a)\lambda(b)}) \leq (r+1.5\epsilon^{1/4})d(\lambda(a), \lambda(b))$*

Proof. Let $m = rd(\lambda(a), \lambda(b))$, and $\{p_j\}_{j=0}^N = \mathcal{S}(\lambda, m)$ so that $d(p_0, p_N) = d(\lambda(a), \lambda(b)) = L$. Let $h_{\overline{p_0 p_N}}$ be the orthogonal projection of λ onto $\overline{p_0 p_N}$, $\tilde{p}_i = h_{\overline{p_0 p_N}}(p_i)$, so $d(\tilde{p}_j, \tilde{p}_{j+1}) \leq d(p_j, p_{j+1}) = m$. Since $\tilde{p}_0 = p_0$, $\tilde{p}_N = p_N$, $\bigcup_{i=0}^{N-1} \overline{\tilde{p}_i \tilde{p}_{i+1}} = \overline{p_0 p_N}$, and Lemma 4.3.1 in the Appendix gives that $d(p_j, \tilde{p}_j) \leq 1.5\epsilon^{1/4}L$. So if $\tilde{p} \in \lambda$, let p_j be the closest point in $\mathcal{S}(\lambda, m)$, we then have $d(\tilde{p}, \overline{p_0 p_N}) \leq d(\tilde{p}, p_j) + d(p_j, \overline{p_0 p_N}) \leq m + 1.5\epsilon^{1/4}L$. Similarly for $\tilde{p} \in \overline{p_0 p_N}$, there is a j such that $\tilde{p} \in \overline{\tilde{p}_j \tilde{p}_{j+1}}$, with $d(\tilde{p}, \tilde{p}_j) \leq d(\tilde{p}, \tilde{p}_{j+1})$, then $d(\tilde{p}, \lambda) \leq d(\tilde{p}, \tilde{p}_j) + d(\tilde{p}_j, \lambda) \leq \frac{1}{2}m + 1.5\epsilon^{1/4}L$. □

The purpose of this subsection is to prove the following lemma which roughly says that given a ϵ , if a path is sufficiently long, then it is ϵ -efficient on some scale.

Lemma 3.2.2. *Let G be a non-degenerate, split abelian-by-abelian group. Take any $N \gg 2$, $L_{stop} \geq (2\kappa)C$, $0 < \epsilon < 1$. If $\tilde{\lambda} : [0, L] \rightarrow G$ is (κ, C) quasi-geodesic satisfying¹*

$$\frac{L_{stop}}{\left(\frac{1}{2}\epsilon^{1/4}\right)^{\frac{h(2\kappa)^2 N + \epsilon}{\epsilon}}} \leq 2\kappa L$$

then there is a scale $0 < \rho_J \leq 1$ such that

$$\frac{|\mathcal{S}(\lambda, \rho_J |\lambda|, \text{ not } \epsilon\text{-efficient at scale } \frac{1}{2}\epsilon^{1/4})|}{|\mathcal{S}(\lambda, \rho_J |\lambda|)|} \leq \frac{1}{N}$$

where $\lambda = \pi_A(\tilde{\lambda})$, and $\frac{1}{2}\epsilon^{1/4}\rho_J |\lambda| \geq L_{stop}$.

Proof. The idea of the proof is as follows: if a segment is not efficient, then by subdividing and adding up the distance between consecutive pairs of points in the subdivision, the sum exceeds the distance between end points of the original segment by a fixed proportion. In other words, lack of efficiency increases length. However this cannot happen at every scale (bigger than $\frac{C}{d(\lambda(0), \lambda(L))}$, where C is the additive constant of the quasi-geodesics), because to every subdivision, the sum of distance between successive pairs of points is bounded above by the length of the curve. We now proceed with the proof.

First note that the condition on L in relation to ϵ , L and N is the same as

$$\frac{\ln(L_{stop}) - \ln(2\kappa L)}{\ln\left(\frac{\epsilon^{1/4}}{2}\right)} - 1 \geq \frac{h(2\kappa)^2}{\epsilon} N \quad (7)$$

If λ is ϵ -efficient at scale $\frac{1}{2}\epsilon^{1/4}|\lambda|$, we can take $r_J = |\lambda|$, $\rho_J = \frac{r_J}{|\lambda|} = 1$ and we are done. Otherwise, let $\{\tilde{p}_j^0\}_{j=0}^{n_0} \subset \{\tilde{p}_j^1\}_{j=0}^{n_1} \subset \{\tilde{p}_j^2\}_{j=0}^{n_2} \cdots \subset \{\tilde{p}_j^D\}_{j=0}^{n_D}$ be an increasing sets of points on $\tilde{\lambda}$ such that

$$(i) \quad r_0 = \frac{1}{2}\epsilon^{1/4}|\lambda|, \quad r_b = \frac{1}{2}\epsilon^{1/4}r_{b-1}, \quad r_D = L_{stop}$$

$$(ii) \quad \{p_j^b\} = \hat{\mathcal{S}}(\lambda, r_b), \text{ where } p_j^b = \pi_A(\tilde{p}_j^b)$$

We note here that for each b between 0 and D , $\lambda_{[p_j^b, p_{j+1}^b]}$ lies in a ball of diameter r_b , we must have $d(\tilde{p}_j^b, \tilde{p}_{j+1}^b) \leq \bar{h}r_b$ by Proposition 3.1.1. Therefore

$$\left| \tilde{\lambda}^{-1}(\tilde{p}_j^b) - \tilde{\lambda}^{-1}(\tilde{p}_{j+1}^b) \right| \leq (\bar{h}r_b)(2\kappa)$$

and

$$\left| \{[\tilde{\lambda}^{-1}(\tilde{p}_j^b), \tilde{\lambda}^{-1}(\tilde{p}_{j+1}^b)]\} \right| \geq \frac{L}{(\bar{h}r_b)(2\kappa)}$$

¹this long expressions really just says that L has to be sufficiently big with respect to given L_{stop} , ϵ and N .

Thus if we denote $\sum_{j=0}^{n_b-1} d(p_j^b, p_{j+1}^b)$ by L_b ,

$$\frac{L_b}{|\lambda|} = \frac{\sum_j d(p_j^b, p_{j+1}^b)}{|\lambda|} \geq \frac{L}{(\hbar r_b)(2\kappa)} r_b \frac{1}{|\lambda|} \geq \frac{L}{\hbar 2\kappa} \frac{1}{2\kappa L} = \frac{1}{\hbar(2\kappa)^2} = \hat{c} \quad (8)$$

which we note is a lower bound that depends only on κ and the group G .

Let E be an integer between 0 and D . By construction, for any p_j^E, p_{j+1}^E , there are s_1, s_2 such that $p_{s_1}^{E+1} = p_j^E, p_{s_2}^{E+1} = p_{j+1}^E$. Then

$$\sum_{i=s_1}^{s_2-1} d(p_i^{E+1}, p_{i+1}^{E+1}) \geq d(p_j^E, p_{j+1}^E)$$

If however the segment of $\lambda_{[p_j^E, p_{j+1}^E]}$ is not ϵ -efficient on scale $\frac{1}{2}\epsilon^{1/4}$, then

$$\sum_{i=s_1}^{s_2-1} d(p_i^{E+1}, p_{i+1}^{E+1}) \geq (1+\epsilon)d(p_j^E, p_{j+1}^E) = d(p_j^E, p_{j+1}^E) + \epsilon d(p_j^E, p_{j+1}^E)$$

This means

$$\sum_{i=0}^{n_{E+1}-1} d(p_i^{E+1}, p_{i+1}^{E+1}) \geq \sum_{j=0}^{n_E-1} d(p_j^E, p_{j+1}^E) + \epsilon \sum_{l \in B_E} d(p_l^E, p_{l+1}^E)$$

where B_E are those integer j between 0 and $n_E - 1$ such that $\lambda_{[p_j^E, p_{j+1}^E]}$ is not ϵ -efficient on scale $\frac{1}{2}\epsilon^{1/4}$. Denote $\sum_{l \in B_E} d(p_l^E, p_{l+1}^E)$ by Ω_E the above says

$$L_{E+1} \geq L_E + (\epsilon)\Omega_E$$

Hence

$$\begin{aligned} |\lambda| \geq L_D &\geq L_{D-1} + (\epsilon)\Omega_{D-1} \\ &\geq L_{D-2} + (\epsilon)\Omega_{D-2} + (\epsilon)\Omega_{D-1} \\ &\geq L_{D-3} + (\epsilon)\Omega_{D-3} + (\epsilon)\Omega_{D-2} + (\epsilon)\Omega_{D-1} \\ &\geq \dots \\ &\geq L_0 + (\epsilon) \sum_{j=1}^D \Omega_{D-j} \geq (\epsilon) \sum_{j=1}^D \Omega_{D-j} \end{aligned}$$

Dividing both sides by $|\lambda|$, and let $\delta_i(\lambda) = \frac{\Omega_i}{L_i}$ be the proportion of elements in $S(\lambda, r_j)$ that are not ϵ -efficient at scale $1/2\epsilon^{1/4}$, we have by equation (8)

$$\frac{1}{\epsilon} \geq \sum_{i=0}^{D-1} \frac{\Omega_i}{|\lambda|} = \sum_{i=0}^{D-1} \frac{\Omega_i}{L_i} \frac{L_i}{|\lambda|} \geq \hat{c} \sum_{i=0}^{D-1} \delta_i(\lambda) \quad (9)$$

Since we stop at $r_D = L_{stop}$,

$$\begin{aligned} \left(\frac{1}{2}\epsilon^{1/4}\right)^{D+1} |\lambda| &= L_{stop} \\ D = \frac{\ln(L_{stop}) - \ln(|\lambda|)}{\ln\left(\frac{1}{2}\epsilon^{1/4}\right)} - 1 &\geq \frac{\ln(L_{stop}) - \ln(2\kappa L)}{\ln\left(\frac{1}{2}\epsilon^{1/4}\right)} \geq \frac{1}{\epsilon \hat{c}} N \end{aligned}$$

where we used equations (7) and (8) in the last inequality.

The right hand side of (9) has at least $\frac{1}{\epsilon c}N$ terms, so for some $0 \leq J \leq D$, $\delta_J(\lambda) \leq \frac{1}{N}$, which means the proportion of segments in $\mathcal{S}(\lambda, r_J)$ that are not ϵ -efficient at scale $\frac{1}{2}\epsilon^{1/4}$ is at most $\frac{1}{N}$. The desired $\rho_J = \frac{r_J}{|\lambda|} = \left(\frac{1}{2}\epsilon^{1/4}\right)^{J+1}$ \square

Corollary 3.2.1. *Let G be a non-degenerate, split abelian-by-abelian group. Take any $2 \ll N_0 < N$, $L_{stop} \geq (2\kappa)C$, $0 < \epsilon < 1$, and let $\mathcal{F} = \{\tilde{\lambda}_i\}$ be a finite set of (κ, C) quasi-geodesics. If every element of \mathcal{F} , $\tilde{\lambda}_i : [0, L_i] \rightarrow G$ satisfies*

$$\frac{L_{stop}}{\left(\frac{1}{2}\epsilon^{1/4}\right)^{\frac{h(2\kappa)^2 N + \epsilon}{\epsilon}}} \leq 2\kappa L_i$$

then there is a scale $0 < \rho_J \leq 1$ and a subset \mathcal{F}_0 such that

$$(i) \quad |\mathcal{F}_0| \geq \left(1 - \frac{N_0}{N}\right)|\mathcal{F}|$$

$$(ii) \quad \text{for every } \tilde{\lambda}_i \in \mathcal{F}_0,$$

$$\frac{|\mathcal{S}(\lambda_i, \rho_J |\lambda_i|, \text{ not } \epsilon \text{ efficient at scale } \frac{1}{2}\epsilon^{1/4})|}{|\mathcal{S}(\lambda_i, \rho_J |\lambda_i|)|} \leq \frac{1}{N_0}$$

where $\lambda_i = \pi_A(\tilde{\lambda}_i)$, and $\frac{1}{2}\epsilon^{1/4}\rho_J |\lambda_i| \geq L_{stop}$.

Proof. We apply Lemma 3.2.2 to each element of \mathcal{F} and stop at equation (9). That is, for every $\tilde{\lambda}_j \in \mathcal{F}$, we have

$$\frac{1}{\epsilon} \geq \hat{c} \sum_{i=0}^{D-1} \delta_i(\lambda_j)$$

therefore

$$\frac{1}{\epsilon} = \frac{1}{|\mathcal{F}|} \sum_{\tilde{\lambda}_j \in \mathcal{F}} \frac{1}{\epsilon} \geq \frac{\hat{c}}{|\mathcal{F}|} \sum_{\tilde{\lambda}_j \in \mathcal{F}} \sum_{i=0}^{D-1} \delta_i(\lambda_j) = \hat{c} \sum_{i=0}^{D-1} \frac{1}{|\mathcal{F}|} \sum_{\tilde{\lambda}_j \in \mathcal{F}} \delta_i(\lambda_j) \quad (10)$$

For the same reason as in Lemma 3.2.2, the right hand side of equation (10) has at least $\frac{1}{\epsilon c}N$ terms, so for some $0 \leq J \leq D$

$$\frac{1}{N} \geq \frac{1}{|\mathcal{F}|} \sum_{\tilde{\lambda}_j \in \mathcal{F}} \delta_J(\lambda_j)$$

Let \mathcal{F}_b be those $\tilde{\lambda}_j \in \mathcal{F}$ whose δ_J value is more than $\frac{1}{N_0}$. Applying Chebyshev inequality we see that

$$\frac{1}{N} \geq |\mathcal{F}_b| \frac{1}{N_0} \frac{1}{|\mathcal{F}|}$$

the claim is obtained by setting \mathcal{F}_0 as the complement of \mathcal{F}_b . \square

The following lemma says that given an efficient segment, most subsegments of length sufficiently larger than the efficient scale are efficient.

Lemma 3.2.3. *Let λ be a rectifiable curve in a metric space Y whose end points are L apart. Suppose λ is ϵ -efficient at scale $\frac{1}{2}\epsilon^{1/4}$. Let $\{q_i\}$ be a subdivision of λ such that for some $r_s, r_b \in [\epsilon^{1/4}L, L]$, the distance between successive subdivision points satisfies $r_s \leq d_Y(q_i, q_{i+1}) \leq r_b$. Then provided $\frac{r_b}{r_s}\epsilon^{1/2} \ll 1$, at least $\epsilon^{1/2}\frac{r_b}{r_s}$ proportion of the subsegments $\{\lambda_{[q_i, q_{i+1}]}\}$ are $\epsilon^{1/2}$ efficient at scale $\frac{1}{2}\epsilon^{1/4}$.*

Proof. Let $\{p_j\} = \bigcup_i \hat{\mathcal{S}}(\lambda_{[q_i, q_{i+1}]}, \frac{1}{2}\epsilon^{1/4}d(q_i, q_{i+1}))$. Then $\{q_j\} \subset \{p_j\}$. Write $q_0 = p_{n_0}$, $q_1 = p_{n_0+n_1}$, $q_2 = p_{n_0+n_1+n_2}$ etc. With this notation we can write

$$\sum_{i=0}^{N-1} d(p_i, p_{i+1}) = \sum_j \sum_{i=n_0+n_1+\dots+n_j}^{n_0+n_1+\dots+n_{j+1}-1} d(p_i, p_{i+1})$$

For each j , write $q_j = p_{s1}$, $q_{j+1} = p_{s2}$. Then **EITHER**

$$d(q_j, q_{j+1}) \leq \sum_{i=0}^{s2-1} d(p_i, p_{i+1}) \leq (1 + \epsilon^{1/2})d(p_{s1}, p_{s2}) = (1 + \epsilon^{1/2})d(q_j, q_{j+1})$$

OR

$$\sum_{i=s1}^{s2-1} d(p_i, p_{i+1}) > (1 + \epsilon^{1/2})d(p_{s1}, p_{s2}) = (1 + \epsilon^{1/2})d(q_j, q_{j+1})$$

in which case we denote the set of all such q_j 's as \mathcal{B} . Note that the cardinality of the coarser division points $|\{q_i\}| \geq \frac{L}{r_b}$.

λ being efficient means

$$\epsilon d(p_0, p_N) \geq \left(\sum_j \sum_{i=n_0+n_1+\dots+n_j}^{n_0+n_1+\dots+n_{j+1}-1} d(p_i, p_{i+1}) \right) - d(p_0, p_N)$$

Hence

$$\begin{aligned} \epsilon L = \epsilon d(p_0, p_N) &\geq \left(\sum_j \sum_{i=n_0+n_1+\dots+n_j}^{n_0+n_1+\dots+n_{j+1}-1} d(p_i, p_{i+1}) \right) - d(p_0, p_N) \\ &\geq \sum_j \left(\sum_{i=n_0+n_1+\dots+n_j}^{n_0+n_1+\dots+n_{j+1}-1} d(p_i, p_{i+1}) - d(q_j, q_{j+1}) \right) \\ &\geq \sum_{q_j \in \mathcal{B}} \epsilon^{1/2} d(q_j, q_{j+1}) \geq \epsilon^{1/2} |\mathcal{B}| r_s \end{aligned}$$

therefore $|\mathcal{B}| \leq \epsilon^{1/2} \frac{L}{r_s}$, giving us a bound on b_r , the proportion of \mathcal{B} , as

$$b_r = \frac{|\mathcal{B}|}{|\{q_j\}|} \leq \frac{\epsilon^{1/2} \frac{L}{r_s}}{\frac{L}{r_b}} = \epsilon^{1/2} \frac{r_b}{r_s}$$

□

3.3 Monotone scale

Definition 3.3.1. (δ -monotone) Let G be a split abelian-by-abelian group, and $\zeta : [0, L] \rightarrow G$ a (κ, C) quasi-geodesic segment such that there exists a line segment $\overline{AB} \in \mathbf{A}$ satisfying $d_H(\pi_A(\zeta), \overline{AB}) \leq \epsilon |\pi_A(\zeta)|$, for some $0 \leq \epsilon < 1$. Let $h_{\overline{AB}} : \pi_A(\zeta) \rightarrow \overline{AB}$ be the map that sends every point of $\pi_A(\zeta)$ to the closest point on \overline{AB} by orthogonal projection. We say that ζ is

- δ -monotone, if $1 > \delta \gg 2\hbar\epsilon$ and

$$h_{\overline{AB}}(\pi_A \circ \zeta(t_1)) = h_{\overline{AB}}(\pi_A \circ \zeta(t_2)) \implies d(\zeta(t_1), \zeta(t_2)) \leq \delta d(\zeta(0), \zeta(L))$$

- (ν, C_1) weakly monotone if for $1 > \nu \gg 2\epsilon\hbar(2\kappa)^2$, $t_1 > t_2$

$$h_{\overline{AB}}(\pi_A \circ \zeta(t_1)) = h_{\overline{AB}}(\pi_A \circ \zeta(t_2)) \implies d(\zeta(t_1), \zeta(t_2)) \leq \nu d(\zeta(t_1), \zeta(0)) + C_1$$

Note that the definition of weakly monotone is not symmetrical to both end points: it's biased towards the starting point $\zeta(0)$.

The following says that in the case of a non-degenerate group, a monotone quasi-geodesic is close to a geodesic segment.

Proposition 3.3.1. Let G be a non-degenerate, split abelian-by-abelian group, and $\lambda : [0, L] \rightarrow G$ a (κ, C) quasi-geodesic whose π_A image is ϵ -efficient. Suppose that with respect to $\lambda(0)$, λ lies outside of the $\frac{3}{\delta d(\lambda(0), \lambda(L))}$ -linear $+C$ neighborhood of the set of walls based at $\lambda(0)$. Then

- (i) λ is within $O(\delta L)^2$ Hausdorff neighborhood of a straight geodesic segment when λ is δ monotone.
- (ii) λ is in $|\Delta|\eta$ -linear $+O(1)^3$ neighborhood of a straight geodesic when λ is (η, C_1) weakly monotone. Recall that Δ is the set of roots of G .

Note that a monotone path is efficient by definition. So being close to a geodesic segment is the same as asking that the movement of the path along \mathbf{H} direction is not too big. We will prove Proposition 3.3.1 by using the observation that for a monotone path in G , admitting a geodesic approximation is the same as saying that for any root $\alpha \in \Delta$, its π_α image admits a (vertical) geodesic approximations. The next lemma sets out one scenario where we have (vertical) geodesic approximation in $H_{n+1} = \mathbb{R}^n \rtimes_\psi \mathbb{R}$. Recall that $\psi(t)$ is $e^t N(t)$, where $N(t)$ is a nilpotent matrix with polynomial entries. We coordinatize points in H_{n+1} as (x, t) , where $x \in \mathbb{R}^n$, and $t \in \mathbb{R}$. Let h denotes for the projection $(x, t) \mapsto t$.

Lemma 3.3.1. Let $\{p_i\}_{i=-s}^t$, where $s, t \in \mathbb{Z}^+$, be points in H_{n+1} such that for some $h_0 > 2$, $h(p_j) = h(p_{j-1}) + h_0$, $\forall j$. For $i > 0$, let d_i denote the distance between p_i and the vertical geodesic passing through p_{i-1} ; for $i < 0$, let d_i denote for the distance between p_i and the vertical geodesic passing p_{i+1} .

- (i) If for all j , $d_j \leq r$, and $2r \ll h_0$, then there is a geodesic γ_0 such that $d(\gamma_0, p_j) \leq 2r$, for all j .

² $O(\delta L)$ here can be taken as $2|\Delta|(\hbar\sqrt{\delta^2 + 4\epsilon^2}|\overline{AB}| + \delta d(\zeta(0), \zeta(L)))$

³ the constant $O(1)$ can be taken as $|\Delta|(\hbar\sqrt{\delta^2 + 4\epsilon^2}|\overline{AB}| + C_1)$

- (ii) If for all j , $d_j \leq \eta|j| + C_1$, where $\eta \ll 1$ and $2C_1 \leq h_0$, then there is a geodesic γ_0 such that $d(\gamma_0, p_j) \leq 2\eta|j| + 2C_1$.

Proof. We first produce geodesic γ^+ and γ^- that stay close to $\{p_i, i \geq 0\}$ and $\{p_i, i \leq 0\}$ respectively. Then we show that γ^+ and γ^- meet at some p_j , $j \geq 0$ and set γ_0 to be the union between $(\gamma^+ \cap \gamma^-)$ and $\gamma^- - (\gamma^+ \cap \gamma^-)$.

Write $p_j = (x_j, t_j)$. We can assume without the loss of generality that $p_0 = (0, 0)$. Note that the distance between a point (x_1, t_1) and the vertical geodesic passing through (x_2, t_2) is $U(|x_1 - x_2|) - t_2$ by equation (2).

- Then, by equation (3), for $j > 0$,

$$\ln |x_j - x_{j-1}| - jh_0 \leq d_j$$

Hence for all $k \geq 0$,

$$|x_k| \leq \sum_{j=1}^k |x_j - x_{j-1}| \leq \sum_{j=1}^k e^{d_j + jh_0}$$

Let γ^+ be the geodesic passing through p_0 . Then for $k \geq 0$,

$$d(p_k, \gamma^+) \leq 2 \ln \left(\sum_{j=1}^k e^{d_j + jh_0} \right) - 2kh_0 = 2 \ln \left(\sum_{j=1}^k e^{d_j + (j-k)h_0} \right)$$

- For $j < 0$, again by equation (3)

$$\ln |x_{j+1} - x_j| - jh_0 \leq d_j$$

Hence

$$|x_{j+1} - x_j| \leq e^{d_j + jh_0}$$

Note that under the assumptions of (i) or (ii), $x_{-\infty} = \lim_{j \rightarrow -\infty} x_j$ exists. So for all $k < 0$,

$$|x_k - x_{-\infty}| \leq \sum_{j=k-1}^{-\infty} e^{d_j + jh_0}$$

Let γ^- be the vertical geodesic passing through $(x_{-\infty}, 0)$. Then for $k < 0$,

$$d(p_k, \gamma^-) \leq 2 \ln \left(\sum_{j=k-1}^{-\infty} e^{d_j + jh_0} \right) - 2kh_0 = 2 \ln \left(\sum_{j=k-1}^{-\infty} e^{d_j + (j-k)h_0} \right)$$

- (i) In this case, $d(p_k, \gamma^+) \leq 2r$ for all $k \geq 0$; $d(p_{k'}, \gamma^-) \leq 2r$ for all $k' \leq 0$. In particular, $d(p_0, \gamma^-) \leq 2r$. Since $\gamma^+ \ni p_0$, the height at which γ^+ and γ^- come together is at most $h(p_0) + 2r < h(p_1)$ by assumption, therefore γ_0 as defined above satisfies the required condition.
- (ii) In this case, $d(p_k, \gamma^+) \leq (2\eta)k + 2C_1$ for $k \geq 0$; $d(p_k, \gamma^-) \leq (2\eta)(-k) + 2C_1$ for $k \leq 0$. In particular, $d(p_0, \gamma^-) \leq 2C_1$, so the height at which γ^+ and γ^- come together occurs no higher than $h(p_0) + 2C_1$. Since $p_0 \in \gamma^+$, γ_0 therefore satisfies the required condition.

□

We now proceed to prove Proposition 3.3.1 by showing that if a path is monotone, then for any root α , its π_α image satisfies the hypothesis of Lemma 3.3.1.

Proof. of Proposition 3.3.1

Set

- $s = \delta|\overline{AB}|$
- $t_j = \max\{t \mid h_{\overline{AB}} \circ \pi_A \circ \lambda(t) = js\}$
- $t'_j = \min\{t \in [t_{j-1}, t_j] \mid h_{\overline{AB}} \circ \pi_A \circ \lambda(t) = js\}$

Therefore for $t \in [t_{j-1}, t'_j]$, we must have $h_{\overline{AB}} \circ \pi_A \circ \lambda(t) \in [(j-1)s, js]$. Since $d(\pi_A(\lambda), \overline{AB}) \leq \epsilon\overline{AB}$, the set $\{\pi_A(\lambda(t)), t \in [t_{j-1}, t'_j]\}$ lies in a ball of diameter at most $\tilde{s} = \sqrt{s^2 + (2\epsilon|\overline{AB}|)^2} = \sqrt{\delta^2 + 4\epsilon^2|\overline{AB}|}$, which means $d(\lambda(t_{j-1}), \lambda(t'_j)) \leq \tilde{h}\tilde{s}$ by Proposition 3.1.1.

(i) In the case that λ is δ monotone,

$$h_{\overline{AB}}(\pi_A \circ \lambda(t_j)) = h_{\overline{AB}}(\pi_A \circ \lambda(t'_j)) \implies d(\lambda(t_j), \lambda(t'_j)) \leq \delta d(\zeta(0), \zeta(L))$$

(ii) If λ is (η, C_1) weakly monotone

$$h_{\overline{AB}}(\pi_A \circ \lambda(t_j)) = h_{\overline{AB}}(\pi_A \circ \lambda(t'_j)) \implies d(\zeta(t_j), \zeta(t'_j)) \leq \eta d(\zeta(t_j), \zeta(0)) + C_1$$

Therefore

$$d(\lambda(t_{j-1}), \lambda(t_j)) \leq d(\lambda(t_{j-1}), \lambda(t'_j)) + d(\lambda(t'_j), \lambda(t_j)) \leq \Upsilon$$

where $\Upsilon = \tilde{h}\tilde{s} + \delta d(\zeta(0), \zeta(L))$ when λ is δ -monotone; and

$\Upsilon = \tilde{h}\tilde{s} + \eta d(\zeta(t_j), \zeta(0)) + C_1$ when λ is (η, C_1) weakly monotone.

By assumption, λ lies outside of $\frac{3}{C}$ -linear $+C$ neighborhood of the set of walls based at $\lambda(0)$. Since $d_H(\pi_A(\lambda), \overline{AB}) \leq \epsilon|\overline{AB}|$, $h(\pi_\Xi \circ \lambda(t_j)) - h(\pi_\Xi \circ \lambda(t_{j-1})) > 2$ for any root Ξ . The claims now follow from application of Lemma 3.3.1 to $\{\pi_\Xi(\lambda(t_j))\}_j$ in the Ξ weight hyperbolic space for each root Ξ . □

We now prove the main lemma in this subsection which roughly says that given $\delta > 0$, a sufficiently long quasi-geodesics whose π_A image is ϵ -efficient, is δ -monotone at some scale.

Lemma 3.3.2. *Let G be a non-degenerate, split abelian-by-abelian group. For any $N \gg 2$, $L_a \geq 2\kappa(C)$, $0 < \delta < 1$, and $\epsilon > 0$, if $\zeta : [0, L] \rightarrow G$ is a (κ, C) quasi-geodesic satisfying*

(i) $\pi_A \circ \zeta$ is ϵ -efficient at scale $\frac{1}{2}\epsilon^{\frac{1}{4}}$, where $\epsilon \leq \min\{(\frac{\delta}{2\tilde{h}})^4, (\frac{\delta}{3.01\tilde{h}})^8, (0.01)^8\}$

(ii) ⁴

$$\frac{\frac{2L_a}{3\epsilon^{1/8}}}{(\delta)^{\frac{(2\kappa)^2 h(2N)}{(1-\epsilon^{1/2}\tilde{h})\delta}}} \leq 2\kappa L$$

⁴this long expressions just says that L is sufficiently big with respect to given data.

then there are scales $\rho_{I+1} < \rho_I \ll 1$ such that for $i = I, I+1$,

$$\frac{|\mathcal{S}(\zeta, \rho_i L, \mathbf{P})|}{|\mathcal{S}(\zeta, \rho_i L)|} \leq \frac{1}{N}$$

where \mathbf{P} is the statement 'either not δ -monotone, or is monotone but of opposite direction to the δ -monotone segment in $\mathcal{S}(\zeta, \rho_{i-1} L)$ to which it is a subset of.

Proof. The idea of the proof is similar to that of Lemma 3.2.2. Suppose the π_A image of a segment is efficient but the segment itself fails to be δ monotone. Then we can find two points whose π_A images are close to each other, but the distance between the two points is very large. By Proposition 3.1.1, this means there must be some point in between those two points whose π_A image is far away from the π_A images of those two points. This means that after a subdivision to the π_A of the segment, the sum of the distance between consecutive points exceeds the distance of its end points by some pre-determined amount. In other words, not monotone gains length. But the length of the π_A image is bounded, a quasi-geodesic cannot fail to be monotone at smaller and smaller scales.

First we note that the conditions on L in relation to L_a , ϵ , δ and N is the same as

$$\frac{\ln\left(\frac{2}{3\epsilon^{1/8}} \frac{L_a}{2\kappa L}\right)}{\ln(\delta)} \geq \frac{(2\kappa)^2 \hbar}{(1 - \epsilon^{1/2} \hbar) \delta} (2N) \quad (11)$$

The conditions on ϵ means that we have

- (I) $2\epsilon^{1/4} \hbar \ll \delta$
- (II) $\epsilon^{1/4} \leq 0.01\epsilon^{1/8}$
- (III) $3.01\epsilon^{1/8} \leq \frac{\delta}{\hbar}$

Write $L_a = d(\pi_A \circ \zeta(0), \pi_A \circ \zeta(L))$. If $\zeta : [0, L] \rightarrow G$ itself is δ monotone, we are done. Otherwise let $\{p_j^0\}_{j=0}^{n_0} \subset \{p_j^1\}_{j=0}^{n_1} \subset \{p_j^2\}_{j=0}^{n_2} \subset \dots \subset \{p_j^D\}_{j=0}^{n_D}$ be an increasing sets of points on ζ such that

- (i) $\{p_j^0\} = \hat{\mathcal{S}}(\zeta, L_1)$, $L_1 = \delta d(\zeta(0), \zeta(L))$
- (ii) For $i \geq 1$, $\{p_j^i\}_{j=0}^{n_i} = \hat{\mathcal{S}}(\zeta, L_{i+1})$, where $L_{i+1} = \delta L_i$. Note $L_{i+1} < L_i$.
- (iii) $1.5(\epsilon^{1/2})^{1/4} L_D = L_a$

Let $0 \leq i \leq D$. $\{\pi_A(p_j^i)\}_{j=0}^{n_i}$ is a subdivision of $\pi_A(\zeta)$. The distance between consecutive points satisfies $\frac{L_{i+1}}{\hbar} \leq d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) \leq L_{i+1}$. We also have $\frac{L_i}{\hbar}, L_i \in [\epsilon^{1/4} L_a, L_a]$. Therefore by Lemma 3.2.3, there is a subset $\mathcal{G}_i \subset \{\pi_A(\zeta)_{[\pi_A(p_j^i), \pi_A(p_{j+1}^i)]}\}$, with $|\mathcal{G}_i| \geq (1 - \epsilon^{1/2} \hbar) |\{\pi_A(\zeta)_{[\pi_A(p_j^i), \pi_A(p_{j+1}^i)]}\}|$, such that whenever $\pi_A(\zeta)_{[\pi_A(p_{j'}^i), \pi_A(p_{j'+1}^i)]} \in \mathcal{G}_i$, it is $\epsilon^{1/2}$ efficient at scale $\frac{1}{2}\epsilon^{1/4}$.

We define the following :

- $\mathcal{C}_i = \{1 \leq j \leq n_i \mid \pi_A(\zeta)_{[\pi_A(p_j^i), \pi_A(p_{j+1}^i)]} \in \mathcal{G}_i\}$ is the set of subsegments produced by $\{p_j^i\}$ whose π_A images are $\epsilon^{1/2}$ efficient at scale $\frac{1}{2}\epsilon^{1/4}$.

- $\mathcal{NC}_i = \{1 \leq j \leq n_i \mid j \notin \mathcal{C}_i\}$, is those subsegments whose π_A images are not $\epsilon^{1/2}$ efficient at scale $\frac{1}{2}\epsilon^{1/4}$. Note $\frac{|\mathcal{C}_i|}{|\mathcal{C}_i|+|\mathcal{NC}_i|} \geq 1 - \epsilon^{1/2}\hbar$
- $\mathcal{B}_i = \{1 \leq j \leq n_i \mid j \in \mathcal{C}_i, \zeta_{[(p_j^i), p_{j+1}^i]} \text{ is not } \delta \text{ monotone}\}$ is those segments whose π_A images are $\epsilon^{1/2}$ efficient at scale $\frac{1}{2}\epsilon^{1/4}$ but fails to be δ monotone.
- $b_i = \frac{|\mathcal{B}_i|}{|\mathcal{C}_i|}$ be the proportion of subsegments that are $\epsilon^{1/2}$ -efficient at scale $1/2\epsilon^{1/4}$ but fails to be δ monotone.
- For $J \in \mathcal{C}_i - \mathcal{B}_i$,
 $\Psi_{i+1,J} = \{j' \in \mathcal{C}_{i+1} - \mathcal{B}_{i+1} \mid \zeta_{[p_{j'}^{i+1}, p_{j'+1}^{i+1}]} \subset \zeta_{[p_J^i, p_{J+1}^i]}\}$
but those two have opposite orientations } are basically those subsegments produced by $\{p_j^{i+1}\}_{j=0}^{n_{i+1}}$ that are δ monotone and belong to a δ monotone subsegment produced by $\{p_j^i\}_{j=0}^{n_i}$ but their orientations do not agree.
- $\mathcal{R}_{i+1} = \bigcup_{J \in \mathcal{C}_i - \mathcal{B}_i} \Psi_{i+1,J}$
- $b_{i+1} = \frac{|\mathcal{R}_{i+1}|}{|\mathcal{C}_{i+1}|}$ be the proportion of subsegments that are $\epsilon^{1/2}$ -efficient at scale $1/2\epsilon^{1/4}$ and δ monotone but of wrong orientation.
- Write $\hat{L} = (1 - \epsilon^{1/2}\hbar)L$ and note that $|\mathcal{C}_i| \geq \frac{\hat{L}}{2\kappa L_{i+1}}$

Since ζ is not δ -monotone, there are two points $t_1, t_2 \in [0, L]$ such that

- (i) $h \frac{d(\pi_A \circ \zeta(t_1), \pi_A \circ \zeta(t_2))}{d(\pi_A \circ \zeta(0), \pi_A \circ \zeta(L))} = h \frac{d(\pi_A \circ \zeta(t_1), \pi_A \circ \zeta(t_2))}{d(\pi_A \circ \zeta(0), \pi_A \circ \zeta(L))}$. This means

$$d(\pi_A \circ \zeta(t_1), \pi_A \circ \zeta(t_2)) \leq 4\epsilon^{1/4}L_a \quad (12)$$

because $\pi_A \circ \zeta$ is ϵ -efficient on scale $\frac{1}{2}\epsilon^{1/4}$, which means the Hausdorff distance between $\pi_A \circ \zeta$ and $\overline{\pi_A \circ \zeta(0)\pi_A \circ \zeta(L)}$ is at most $2\epsilon^{1/4}L_a$.

AND

- (ii) $d(\zeta(t_1), \zeta(t_2)) \geq \delta d(\zeta(0), \zeta(L))$. By Proposition 3.1.1, this means $\exists t \in [t_1, t_2]$ such that

$$d(\pi_A \circ \zeta(t), \pi_A \circ \zeta(t_i)) \geq \frac{\delta}{h} d(\zeta(0), \zeta(L)) \quad (13)$$

for $i = 1, 2$ in light of (12)

Equations (12) and (13) together means

$$\sum_{j=0}^{n_0} d(\pi_A(p_j^0)b, \pi_A(p_{j+1}^0)) - d(\pi_A(p_0^0), \pi_A(p_{n_0}^0)) \geq \left(2\frac{\delta}{h}d(\zeta(0), \zeta(L)) - 4\epsilon^{1/4}L_a\right)$$

i.e.

$$\sum_{j=0}^{n_0} d(\pi_A(p_j^0)b, \pi_A(p_{j+1}^0)) - L_a \geq \left(\frac{2\delta}{h} - 4\epsilon^{1/4}\right)L_a \gg 0$$

where we used property (I) for the last inequality and recalled that $L_a = d(\pi_A(p_0^0), \pi_A(p_{n_0}^0))$.

Now for each $D \geq i \geq 1$, $1 \leq j \leq n_i$,

- EITHER $j \in \mathcal{B}_i$. In this case $\pi_A \circ \zeta_{[p_j^i, p_{j+1}^i]}$ is $\epsilon^{1/2}$ efficient at scale $\frac{1}{2}\epsilon^{1/4}$ but not δ -monotone.

Then there are two points $t_1, t_2 \in [\zeta^{-1}(p_j^i), \zeta^{-1}(p_{j+1}^i)]$ such that

$$(i) \quad h_{\pi_A(p_j^i), \pi_A(p_{j+1}^i)}(\pi_A \circ \zeta(t_1)) = h_{\pi_A(p_j^i), \pi_A(p_{j+1}^i)}(\pi_A \circ \zeta(t_2))$$

This means that

$$d(\pi_A \circ \zeta(t_1), \pi_A \circ \zeta(t_2)) \leq 2 \left(\frac{3}{2} (\epsilon^{\frac{1}{2}})^{\frac{1}{4}} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \frac{1}{2} \epsilon^{1/4} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) \right)$$

by Lemma 3.2.1.

Therefore by property (II) in the hypothesis

$$d(\pi_A \circ \zeta(t_1), \pi_A \circ \zeta(t_2)) \leq 3.01 \epsilon^{1/8} L_{i+1} \quad (14)$$

AND

- (ii) $d(\zeta(t_1), \zeta(t_2)) > \delta L_{i+1}$. By Proposition 3.1.1, this means $\exists t \in [t_1, t_2]$ such that

$$d(\pi_A(\zeta(t)), \pi_A(\zeta(t_i))) \geq \frac{\delta}{h} L_{i+1} \quad (15)$$

for $i = 1, 2$ in light of (14)

say $p_j^i = p_{s1_j}^{i+1}$, $p_{j+1}^i = p_{s2_j}^{i+2}$, then (14) together with 15 imply that

$$\begin{aligned} \sum_{t=s1_j}^{s2_j-1} d(\pi_A(p_t^{i+1}), \pi_A(p_{t+1}^{i+1})) &\geq d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \left(2 \frac{\delta L_{i+1}}{h} \right) - 3.01 \epsilon^{1/8} L_{i+1} \\ &\geq d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + H_{i+1} \delta L_{i+1} \end{aligned}$$

where we have set constants H_{i+1} to satisfy

$$\frac{2\delta}{h} - 3.01 \epsilon^{1/8} \geq H_{i+1} \delta \quad (16)$$

Summing over all $j \in \mathcal{B}_i$ we have

$$\begin{aligned}
\sum_{j \in \mathcal{B}_i} \sum_{t=s_{1_j}^{s_{2_j}-1}} d(\pi_A(p_t^{i+1}), \pi_A(p_{t+1}^{i+1})) &\geq \sum_{j \in \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + |\mathcal{B}_i| H_{i+1} \delta L_{i+1} \\
&\geq \sum_{j \in \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \mathfrak{b}_i |\mathcal{C}_i| H_{i+1} \delta L_{i+1} \\
&\geq \sum_{j \in \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \mathfrak{b}_i \frac{\hat{L}}{2\kappa L_{i+1}} H_{i+1} \delta L_{i+1}
\end{aligned}$$

That is,

$$\sum_{j \in \mathcal{B}_i} \sum_{t=s_{1_j}^{s_{2_j}-1}} d(\pi_A(p_t^{i+1}), \pi_A(p_{t+1}^{i+1})) \geq \sum_{j \in \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \frac{\mathfrak{b}_i \hat{L}}{2\kappa} H_{i+1} \delta \quad (17)$$

- OR $j \in \mathcal{C}_i - \mathcal{B}_i$. In this case $\zeta_{[p_j^i, p_{j+1}^i]}$ is δ -monotone. Write $p_j^i = p_{s_{1_j}}^{i+1}$, $p_{j+1}^i = p_{s_{2_j}}^{i+1}$, then

$$\sum_{t=s_{1_j}^{s_{2_j}-1}} d(\pi_A(p_t^{i+1}), \pi_A(p_{t+1}^{i+1})) - \sum_{z \in \Psi_{i+1,j}} \frac{L_{i+2}}{\hbar} \geq d(\pi_A(p_j^i), \pi_A(p_{j+1}^i))$$

Summing over all $j \in \mathcal{C}_i - \mathcal{B}_i$ we have

$$\begin{aligned}
\sum_{j \in \mathcal{C}_i - \mathcal{B}_i} \sum_{t=s_{1_j}^{s_{2_j}-1}} d(\pi_A(p_t^{i+1}), \pi_A(p_{t+1}^{i+1})) &\geq \sum_{j \in \mathcal{C}_i - \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \sum_{j \in \mathcal{C}_i - \mathcal{B}_i} \sum_{z \in \Psi_{i+1,j}} \frac{L_{i+2}}{\hbar} \\
&= \sum_{j \in \mathcal{C}_i - \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + |\mathcal{R}_{i+1}| \frac{L_{i+2}}{\hbar} \\
&\geq \sum_{j \in \mathcal{C}_i - \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \mathfrak{b}_{i+1} |\mathcal{C}_{i+1}| \frac{L_{i+2}}{\hbar} \\
&\geq \sum_{j \in \mathcal{C}_i - \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \mathfrak{b}_{i+1} \frac{\hat{L}}{2\kappa L_{i+2}} \frac{L_{i+2}}{\hbar}
\end{aligned}$$

That is,

$$\sum_{j \in \mathcal{C}_i - \mathcal{B}_i} \sum_{t=s_{1_j}^{s_{2_j}-1}} d(\pi_A(p_t^{i+1}), \pi_A(p_{t+1}^{i+1})) \geq \sum_{j \in \mathcal{C}_i - \mathcal{B}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) + \frac{\mathfrak{b}_{i+1} \hat{L}}{2\kappa \hbar} \quad (18)$$

- OR $j \in \mathcal{N}\mathcal{C}_i$. Write $p_j^i = p_{s_{1_j}}^{i+1}$, $p_{j+1}^i = p_{s_{2_j}}^{i+1}$, then

$$\sum_{t=s_{1_j}^{s_{2_j}-1}} d(\pi_A(p_t^{i+1}), \pi_A(p_{t+1}^{i+1})) \geq d(\pi_A(p_j^i), \pi_A(p_{j+1}^i))$$

Summing over all $j \in \mathcal{NC}_i$ we have

$$\sum_{j \in \mathcal{NC}_i} \sum_{t=s1_j}^{s2_j-1} d(\pi_A(p_t^{i+1}), \pi_A(p_t^{i+1})) \geq \sum_{j \in \mathcal{NC}_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i)) \quad (19)$$

Putting (17), (18) and (19) together, we have

$$\sum_{1 \leq w \leq n_{i+1}} d(\pi_A(p_w^{i+1}), \pi_A(p_{w+1}^{i+1})) \geq \sum_{1 \leq z \leq n_i} d(\pi_A(p_z^i), \pi_A(p_{z+1}^i)) + \frac{b_i \hat{L}}{2\kappa} H_{i+1} \delta + \frac{\mathfrak{b}_{i+1} \hat{L}}{2\kappa}$$

By equation (16), H_i satisfies

$$\frac{2\delta}{h} - 3.01\epsilon^{1/8} \geq H_{i+1} \delta$$

by property (III) in the hypothesis on ϵ and δ , $3.01\epsilon^{1/8} \leq \frac{\delta}{h}$, so we can take $H_i = \frac{1}{h}$, hence

$$\sum_{1 \leq w \leq n_{i+1}} d(\pi_A(p_w^{i+1}), \pi_A(p_{w+1}^{i+1})) \geq \sum_{1 \leq z \leq n_i} d(\pi_A(p_z^i), \pi_A(p_{z+1}^i)) + b_i \frac{\delta \hat{L}}{2\kappa h} + \mathfrak{b}_{i+1} \frac{\hat{L}}{2\kappa} \quad (20)$$

Write $\lambda_i = \sum_{1 \leq j \leq n_i} d(\pi_A(p_j^i), \pi_A(p_{j+1}^i))$, and using (20) we have

$$\begin{aligned} 2\kappa L \geq \lambda_D - L_a &= \left(\sum_{i=0}^{D-1} \lambda_{i+1} - \lambda_i \right) + \lambda_0 - L_a \geq \sum_{i=0}^{D-1} \lambda_{i+1} - \lambda_i \\ &\geq \sum_{i=0}^{D-1} \frac{b_i \delta \hat{L}}{2\kappa h} + \frac{\mathfrak{b}_{i+1} \hat{L}}{2\kappa} \\ &= \frac{\delta \hat{L}}{2\kappa h} \sum_{i=0}^{D-1} b_i + \frac{\hat{L}}{2\kappa} \sum_{i=0}^{D-1} \mathfrak{b}_{i+1} \end{aligned}$$

Divide both sides by $\hat{L} = (1 - \epsilon^{1/2} h) L$

$$\frac{2\kappa}{1 - \epsilon^{1/2} h} \geq \frac{\delta}{2\kappa h} \sum_{i=0}^{D-1} b_i + \frac{1}{2\kappa} \sum_{i=0}^{D-1} \mathfrak{b}_{i+1} \geq \frac{\delta}{2\kappa h} \sum_{i=0}^{D-1} (b_i + \mathfrak{b}_{i+1}) \quad (21)$$

Since $L_{i+1} = \delta L_i$ for $i \geq 1$, $L_1 = \delta d(\zeta(0), \zeta(L))$, $L_i = \delta^i d(\zeta(0), \zeta(L))$. The condition on L_D means

$$\begin{aligned} 1.5(\epsilon^{1/2})^{1/4} L_D &= L_a \\ 1.5\epsilon^{1/8} \delta^D d(\zeta(0), \zeta(L)) &= L_a \\ \frac{2}{3\epsilon^{1/8}} \frac{L_a}{d(\zeta(0), \zeta(L))} &= \delta^D \\ D &= \frac{\ln(\frac{2}{3\epsilon^{1/8}} \frac{L_a}{d(\zeta(0), \zeta(L))})}{\ln(\delta)} \end{aligned}$$

By equation(11), we have $D \geq 2N \frac{(2\kappa)^2 \hbar}{\delta(1-\epsilon^{1/2}\hbar)}$. So equation (21) implies that for some $1 \leq I \leq D-1$ we must have

$$\flat_{I-1} + \natural_{I-1} \leq \frac{1}{N}, \quad \flat_I + \natural_I \leq \frac{1}{N}$$

Recall that $\flat_{I'}$ is the proportion of efficient subsegments produced by $\{p_j^{I'}\}_{j=0}^{n_{I'}}$ that are not monotone, and $\natural_{I'}$ is the proportion that are monotone but of the wrong orientation. The desired $\rho_I = \frac{L_I}{L}$, $\rho_{I+1} = \frac{L_{I+1}}{L}$ \square

Corollary 3.3.1. *Let G be a non-degenerate, split abelian-by-abelian group. Take any $2 \ll N_0 < N/2$, $L_0 \geq 2\kappa(C)$, $0 < \delta < 1$, and $\epsilon > 0$, and let $\mathcal{F} = \{\zeta_j\}$ be a finite set of (κ, C) quasi-geodesics. If every element of \mathcal{F} , $\zeta_j : [0, L_j] \rightarrow G$ satisfies the following:*

(i) $\pi_A \circ \zeta_j$ is ϵ -efficient at scale $\frac{1}{2}\epsilon^{\frac{1}{4}}$, where $\epsilon \leq \min\{(\frac{\delta}{2\hbar})^4, (\frac{\delta}{3.01\hbar})^8, (0.01)^8\}$

(ii)

$$\frac{\frac{2L_0}{3\epsilon^{1/8}}}{(\delta) \frac{(2\kappa)^2 \hbar (2N)}{(1-\epsilon^{1/2}\hbar)\delta}} \leq 2\kappa L_j$$

then there are scales $\rho_{I+1} < \rho_I \ll 1$, and a subset $\mathcal{F}_0 \subset \mathcal{F}$ such that

(i) $|\mathcal{F}_0| \geq (1 - \frac{2N_0}{N}) |\mathcal{F}|$

(ii) for every $\zeta \in \mathcal{F}_0$, and $i = I, I+1$,

$$\frac{|\mathcal{S}(\zeta, \rho_i L, \mathbf{P})|}{|\mathcal{S}(\zeta, \rho_i L)|} \leq \frac{1}{N}$$

where \mathbf{P} is the statement 'either not δ -monotone, or is monotone but of opposite direction to the δ -monotone segment in $\mathcal{S}(\zeta, \rho_{i-1} L)$ to which it is a subset of.

Proof. We apply Lemma 3.3.2 to each element of \mathfrak{G} since its elements are all (κ, C) quasi-geodesics. We arrive at equation (21) for each element of \mathcal{F} . That is,

$$\frac{2\kappa}{1-\epsilon^{1/2}\hbar} \geq \frac{\delta}{2\kappa\hbar} \sum_{i=0}^{D-1} (\flat_i(\zeta_j) + \natural_{i+1}(\zeta_j))$$

therefore

$$\frac{2\kappa}{1-\epsilon^{1/2}\hbar} \geq \frac{1}{|\mathcal{F}|} \sum_{\zeta_j \in \mathcal{F}} \frac{\delta}{2\kappa\hbar} \sum_{i=0}^{D-1} (\flat_i(\zeta_j) + \natural_{i+1}(\zeta_j)) = \frac{\delta}{2\kappa\hbar} \sum_{i=0}^{D-1} \frac{1}{|\mathcal{F}|} \sum_{\zeta_j \in \mathcal{F}} (\flat_i(\zeta_j) + \natural_{i+1}(\zeta_j))$$

Counting the number of terms on the right hand side means that for some $1 \leq I \leq D-1$,

$$\begin{aligned} \frac{1}{|\mathcal{F}|} \sum_{\zeta_j \in \mathcal{F}} (\flat_{I-1}(\zeta_j) + \natural_{I-1}(\zeta_j)) &\leq \frac{1}{N} \\ \frac{1}{|\mathcal{F}|} \sum_{\zeta_j \in \mathcal{F}} (\flat_I(\zeta_j) + \natural_I(\zeta_j)) &\leq \frac{1}{N} \end{aligned}$$

Let \mathcal{F}_b consist of those ζ whose $b_I + \mathfrak{b}_I$ or $b_{I-1} + \mathfrak{b}_{I-1}$ values is more than $\frac{1}{N_0}$. The desired claim is obtained after applying Chebyshev inequality and setting \mathcal{F}_0 as the complement of \mathcal{F}_b , and $\rho_I = \delta^I$, $\rho_{I+1} = \delta^{I+1}$. \square

3.4 Occurrence of weakly monotone segments

In the previous subsection we showed the existence of a δ -monotone scale. In this subsection, we write G for a non-degenerate, split abelian-by-abelian group, and we will see that by chaining a lot of δ -monotone segments together, we end up with a path that is weakly monotone.

Definition 3.4.1. *Let G be a non-degenerate, split abelian-by-abelian group. Let $\zeta : [0, L] \rightarrow G$ be a (κ, C) quasi-geodesic segment that is δ -monotone. Suppose for some $L_s \gg 2\kappa C$,*

$$\frac{|\mathcal{S}(\zeta, L_s, \mathbf{P})|}{|\mathcal{S}(\zeta, L_s)|} \leq \frac{1}{N}$$

where \mathbf{P} is the statement "not δ monotone, or is δ monotone but with opposite orientation from ζ ".

For a point $x \in \zeta$, define

$$P(x, \zeta, T) = |B(x, T) \cap \Delta|, \text{ where } \Delta = \bigcup_{\lambda \in \mathcal{S}(\zeta, L_s, \mathbf{P})} \lambda$$

We say x is (M) uniform along ζ if for all $T \geq 0$,

$$P(x, \zeta, T) \leq M \left(\frac{|\mathcal{S}(\zeta, L_s, \mathbf{P})|}{|\mathcal{S}(\zeta, L_s)|} \right) T$$

where M satisfies $\frac{M}{N} \ll 1$.

The main lemma of this subsection is:

Lemma 3.4.1. *Let ζ be a δ -monotone (κ, C) quasi-geodesic in G . Suppose $x \in \zeta$ is a uniform point, with $P(x, \zeta, T) \leq \nu T$, $\nu \ll 1$. Then ζ consider as a (κ, C) quasi-geodesic leaving x at $T = 0$ is $(\nu(1 + \hbar), 2\kappa L_s)$ weakly monotone.*

Proof. let h denote the projection of $\pi_A(\zeta)$ onto the straight line joining the end points of $\pi_A(\zeta)$.

Then up to time T , provided $T > L_s$, at most ν proportion of segments in $\mathcal{S}(\zeta, L_s)$ belong to $\mathcal{S}(\zeta, L_s, \mathbf{P})$, and at least $1 - \nu$ proportion are δ monotone. Therefore

$$h(\zeta(T)) - h(x) \geq \frac{(1 - \nu)T}{\hbar} - \nu T = (1 - \nu - \hbar\nu) \frac{T}{\hbar} \quad (22)$$

so $\pi_A(\zeta)$ moves at a linear rate.

For any $\hat{s} > 0$, let t_1, t_2 be the smallest and largest number t such that $h(\zeta(t)) = \hat{s}$. Let b denotes the proportion of $\mathcal{S}(\zeta, L_s)$ in between $\zeta(t_1)$ and $\zeta(t_2)$ that belongs to $\mathcal{S}(\zeta, L_s, \mathbf{P})$. Either $\zeta(t_2) - \zeta(t_1) \leq L_s$, in which case $t_2 - t_1 \leq 2\kappa L_s$; OR $\zeta(t_2) - \zeta(t_1) > L_s$, in which case we have

$$0 = h(\zeta(t_2)) - h(\zeta(t_1)) \geq \frac{(1 - b)(t_2 - t_1)}{\hbar} - b(t_2 - t_1)$$

which means $b > \frac{1}{1+\hbar}$. On the other hand, we also know that $b(t_2 - t_1) \leq \nu t_2$, therefore

$$t_2 - t_1 \leq \frac{\nu t_2}{b} \leq (1 + \hbar)\nu t_2$$

That is, whenever $h(t_2) = h(t_1)$, we must have $t_2 - t_1 \leq (1 + \hbar)\nu t_2 + 2\kappa L_s$. \square

The following lemma provides us with abundant supply of uniform points.

Lemma 3.4.2. *Let ζ be a δ -monotone (κ, C) quasi-geodesic in G . Then the proportion of non- M uniform points in $\hat{\mathcal{S}}(\zeta, L_s)$ is at most $\frac{2}{M}$.*

Proof. Write Δ as the union of all segments in $\mathcal{S}(\zeta, L_s, \mathbf{P})$, N as the measure of the union of all the segments in $\mathcal{S}(\zeta, L_s)$, and $\mu = \frac{|\Delta|}{N}$. For every non-uniform point x , we can find an interval I_x , such that

$$|I_x \cap \Delta| \geq M\mu|I_x|$$

Then the collection of all such interval $\{I_x\}$ forms a cover for the set of non-uniform points. Choose a subcover so that $\sum |I_x \cap \Delta| \leq 2|\Delta|$. Then

$$\left| \bigcup I_x \right| \leq \sum |I_x| \leq \sum \frac{1}{M\mu} |I_x \cap \Delta| \leq \frac{2|\Delta|}{M\mu}$$

now divide both sides by N . \square

Remark 3.4.1. *Let ζ be a quasi-geodesic segment that lies within (ν, c) -linear neighborhood of a geodesic segment, where $c \ll \nu|\zeta|$. In light of Lemma 3.4.1, we may call a point $p \in \zeta$ as a ν -uniform point if the subsegments of ζ of length $\gg \nu|\zeta|$, viewed as quasi-geodesics starting from p , lies in (ν, c') -linear neighborhood of geodesic segments for some $c' \ll \nu|\zeta|$.*

Remark 3.4.2. *By abuse of notation, from now on, when we say a point p is M uniform with respect to a quasi-geodesic segment for some $M \gg 1$, we mean definition 3.4.1; if we say p is ν uniform, where $\nu < 1$, we mean remark 3.4.1.*

3.5 Proof of Theorem 3.1

So far our results from previous sections only require the group to be non-degenerate and split abelian-by-abelian. From now on, we will require all our groups to be unimodular.

Proposition 3.5.1. *Let G, G' be non-degenerate, unimodular, split abelian-by-abelian Lie groups, and $\phi : G \rightarrow G'$ be a (κ, C) quasi-isometry. Then, to any $0 < \delta, \eta < \tilde{\eta} < 1$, there are numbers L_0 , $m > 1$, and $0 < \rho_s < \rho_b < \rho_b \leq 1$ depending on δ, η , and κ, C , with the following properties:*

If $\Omega \subset \mathbf{A}$ is a product of intervals of equal size at least mL_0 , by writing

- $\mathcal{P} = \phi(\mathcal{P}(\Omega))$ *as the ϕ images of points in $\mathbf{B}(\Omega)$,*
- $\mathbf{L} = \bigcup_{\zeta \in \phi(\mathcal{L}(\Omega)[m])} \mathcal{S}(\zeta, \hbar\rho_s\rho_b|\zeta|)$ *as the union of subsegments obtained by dividing each $\zeta \in \mathcal{L}(\Omega)[m]$ at scale $\hbar\rho_s\rho_b$.*

Then,

(i) there is a subset $\mathbf{L}_0 \subset \mathbf{L}$, with $|\mathbf{L}_0| \geq (1 - \tilde{q})|\mathbf{L}|$,

(ii) there is also a subset $\tilde{\mathcal{P}} \subset \mathcal{P}$, with $|\tilde{\mathcal{P}}| \geq (1 - Q)|\mathcal{P}|$

such that for every $p \in \tilde{\mathcal{P}}$, amongst all elements in \mathbf{L} containing p , at least $1 - \tilde{Q}$ proportion of them belong to \mathbf{L}_0 , and of those, a further $1 - \hat{Q}$ proportion admit geodesic approximation. That is, if γ is in this set, then it is within $(\eta, (\delta + \frac{\rho_{b'}}{\rho_b})|\gamma|)$ -linear neighborhood of a geodesic segment that makes an angle of at least $\sin^{-1}(\tilde{\eta})$ with root kernel directions. Here $\tilde{\eta}, \tilde{q}, Q, \tilde{Q}, \hat{Q} \rightarrow 0$ as $\eta, \delta, \tilde{\eta}$ approach zero.

Proof. Recall that $G = \mathbf{H} \rtimes \mathbf{A}$, $G' = \mathbf{H}' \rtimes \mathbf{A}'$, where $\mathbf{H}', \mathbf{H}, \mathbf{A}', \mathbf{A}$ are all abelian, and Δ' is the set of roots of G' . First, we choose the following constants:

- $N \gg \dim(\mathbf{A}')$ such that $\frac{|\Delta'|}{\sqrt{N}}(1 + \hbar) < \eta$,
- $m = N^{1/3}$
- $\epsilon \leq \tilde{\eta} \min\{(\frac{\delta}{2\hbar})^4, (\frac{\delta}{3.01\hbar})^8, (0.01)^8\}$
- $L_a \geq \frac{3}{\epsilon}$
- L_{stop} such that

$$\frac{\frac{2L_a}{3\epsilon^{1/8}}}{\frac{(2\kappa)^2 \hbar (2N)}{(\delta)^{(1-\epsilon^{1/2}\hbar)\delta}}} \leq 2\kappa L_{stop}$$

- L_0 such that

$$\frac{L_{stop}}{\left(\frac{1}{2}\epsilon^{1/4}\right)^{\frac{\hbar(2\kappa)^2 N + \epsilon}{\epsilon}}} \leq 2\kappa L_0$$

- $M = \eta\sqrt{N}$.

Let $\Omega \subset \mathbf{A}$ be a product of intervals of equal size at least mL_0 . We will build $\tilde{\mathcal{P}}$ from the ϕ images of $\mathcal{P}(\Omega) = \bigcup_{\zeta \in \mathcal{L}(\Omega)[m]} \mathcal{P}(\zeta)$.

Let $\mathcal{F} = \phi(\mathcal{L}(\Omega)[m])$, and apply Lemma 10 to \mathcal{F} and $N_0 = \sqrt{N}$ to obtain a scale ρ_J and subset \mathcal{F}_0 such that

$$\left| \bigcup_{\zeta \in \mathcal{F}_0} \mathcal{S}(\pi_A(\zeta), \rho_J |\pi_A \circ \zeta|, \epsilon\text{-efficient at scale } 1/2\epsilon^{1/4}) \right| \geq \left(1 - \frac{1}{\sqrt{N}}\right)^2 \left| \bigcup_{\zeta \in \mathcal{F}} \mathcal{S}(\pi_A(\zeta), \rho_J |\pi_A \circ \zeta|) \right|$$

Write \mathfrak{M} for the union of $\mathcal{S}(\pi_A(\zeta), \rho_J |\pi_A \circ \zeta|)$ as ζ ranges over \mathcal{F} , and \mathfrak{M}_0 for the subset of \mathfrak{M} that are ϵ -efficient at scale $1/2\epsilon^{1/4}$. The above equation says $|\mathfrak{M}_0| \geq (1 - 1/\sqrt{N})^2 |\mathfrak{M}|$.

Each element of \mathfrak{M}_0 is the π_A image of a subsegment of $\phi(\mathcal{L}(\Omega)[m])$. Let \mathcal{G} be the π_A pre-images of \mathfrak{M}_0 . That is, $\zeta \in \mathcal{G}$ means $\pi_A(\zeta) \in \mathfrak{M}_0$. We now apply Lemma 3.3.1 to \mathcal{G} , and again taking $N_0 = \sqrt{N}$, to obtain scales ρ_I, ρ_{I+1} and a subset \mathcal{G}_0 such that

$$\left| \bigcup_{\gamma \in \mathcal{G}_0} \mathcal{S}(\gamma, \rho_I |\gamma|, \delta\text{-monotone}) \right| \geq \left(1 - \frac{1}{\sqrt{N}}\right)^2 \left| \bigcup_{\gamma \in \mathcal{G}} \mathcal{S}(\gamma, \rho_I |\gamma|) \right|$$

In other words, setting \mathbf{L} as the union of $\mathcal{S}(\zeta, \hbar \rho_I \rho_J |\zeta|)$ where ζ ranges over all \mathcal{F} , we have obtained a subset \mathbf{L}_g whose measure is at least $(1 - 1/\sqrt{N})^4$ that of \mathbf{L} , and each element in \mathbf{L}_g is δ monotone.

Recall that $\mathcal{P} = \phi(\mathcal{P}(\Omega)) = \bigcup_{\zeta \in \mathbf{L}} \mathcal{P}(\zeta)$, and for those $p \in \mathcal{P}(\Omega)$ such that $d(p, \partial \mathbf{B}(\Omega)) \geq L_a/(2\kappa)$, the intersection between the union of elements in \mathcal{F} and $B(\phi(p), L_a)$ has full measure. Since $\mathbf{B}(\Omega)$ has small boundary area compared to its volume, and the ratio of L_a to L_0 is a function of δ that goes to zero as δ approaches zero, we have a subset $\mathbf{L}_0 \subset \mathbf{L}_g$ with relative measure at least $1 - \vartheta$ whose elements make an angle at least $\sin^{-1}(\tilde{\eta})$ with root kernels. Here ϑ goes to zero as $\tilde{\eta}$ and δ approach zero.

Let $\mathcal{P}_g \subset \mathcal{P}$ be those images coming from a point in $\mathcal{P}(\Omega)$ at least $L_a/(2\kappa)$ away from $\partial \mathbf{B}(\Omega)$. We will extract those points of \mathcal{P}_g that are M uniform with respect to at least s proportion of those elements in \mathbf{L}_0 that contain it. We will then choose s appropriately so that the relative proportion of $\mathcal{P} - \mathcal{P}_0$ is small and depends on our input data.

To begin, we note that the incident relation between \mathcal{P}_g and \mathbf{L}_0 is symmetrical. Moreover we know that for any two points in \mathcal{P}_g , the ratio of numbers of elements in \mathbf{L}_0 containing each of them is bounded by $2^{\dim(\mathbf{A})^\dagger}$. For any two elements of \mathbf{L} , the ratio of numbers of points in \mathcal{P}_g lying on each of them is bounded by m .

For $p \in \mathcal{P}_g$ (resp. $\zeta \in \mathbf{L}_0$) write $\mathbf{Y}(p)$ (resp. $\mathcal{P}(\zeta)$) for the set of elements in \mathbf{L}_0 (resp. \mathcal{P}_g) incident with p (resp. ζ). Let $\mathcal{BP} \subset \mathcal{P}_g$ consisting of points that fails to be M -uniform with respect to at least s proportion of elements in $\mathbf{Y}(p)$.

We know that for $\zeta \in \mathbf{L}_0$, the proportion of non M -uniform points is at most $\frac{2}{M}$. Let χ denote for the characteristic function of the subset of $\{(p, \zeta) : p \in \mathcal{P}_g, \zeta \in \mathbf{L}_0, p \in \zeta\}$ consisting of pairs (p, ζ) such that p fails to be M -uniform of ζ . Then, starting from

$$\sum_{p \in \mathcal{P}_g} \sum_{\zeta \in \mathbf{Y}(p)} \chi = \sum_{\zeta \in \mathbf{L}_0} \sum_{p \in \zeta} \chi$$

we have

$$s |\mathbf{Y}(p)|_{\min} |\mathcal{BP}| \leq \sum_{p \in \mathcal{BP}} \sum_{\zeta \in \mathbf{Y}(p)} \chi = \sum_{p \in \mathcal{P}_g} \sum_{\zeta \in \mathbf{Y}(p)} \chi$$

and

$$\sum_{\zeta \in \mathbf{L}_0} \sum_{p \in \zeta} \chi \leq \sum_{\zeta \in \mathbf{L}_0} \frac{2}{M} |\mathcal{P}(\zeta)| \leq \frac{2}{M} |\mathbf{L}_0| |\mathcal{P}(\zeta)|_{\max}$$

Therefore $|\mathcal{BP}| \leq \frac{2/M}{s} 2^{\dim(\mathbf{A})} m |\mathcal{P}_g|$. By choosing $s = \frac{2}{N^{1/6}} 2^{\dim(\mathbf{A})} m$, we have $\frac{|\mathcal{BP}|}{|\mathcal{P}|} \leq \frac{\tilde{\eta}}{N^{1/3}}$. Setting \mathcal{P}_0 as $\mathcal{P}_g - \mathcal{BP}$. The desired claim now follows after Lemma 3.4.1. \square

We can now prove Theorem 3.1.

Proof. Theorem 3.1 We apply Proposition 3.5.1 to $\mathcal{L}(\Omega)$ to obtain two scales: $\varrho_1 = \hbar \rho_s \rho_{b'}$, $\varrho_2 = \hbar \rho_s \rho_b$, and a subset $\tilde{\mathbf{L}}_0 \subset \mathbf{L} = \bigcup_{\gamma \in \mathcal{L}(\Omega)} \mathcal{S}(\phi(\gamma), \varrho_2 |\phi(\gamma)|)$, such that if $\zeta \in \tilde{\mathbf{L}}_0$, then ζ is within $(\eta, \frac{\varrho_1}{\varrho_2} |\zeta|)$ -linear neighborhood of a geodesic segment that makes an angle of at least $\sin^{-1}(\eta)$ with root kernel directions.

[†]because Ω is a product of intervals

For each $\gamma \in \mathcal{L}(\Omega)$, the pre-images of $\mathcal{S}(\phi(\gamma), \varrho_2|\phi(\gamma)|)$ under ϕ are subsegments $C_\gamma = \{\gamma_i\}$ whose union is γ , whose lengths lie between $\frac{\varrho_2}{2\kappa}|\gamma|$ and $2\kappa\varrho_2|\gamma|$. Furthermore, the subset $C_\gamma^0 = \{\zeta \in C_\gamma : \phi(\zeta) \in \tilde{\mathbf{L}}_0\}$ has large measure. If for some $\gamma_i \in C_\gamma^0$, an element $\zeta \in \mathcal{S}(\gamma, \frac{\varrho_2}{2\kappa}|\gamma|)$ satisfies $|\zeta \cap \gamma_i| \geq (1 - \frac{\varrho_1}{\varrho_2 2\kappa})|\zeta|$, then $\phi(\zeta)$ is within $(\eta, \frac{\varrho_1}{\varrho_2}|\phi(\zeta)|)$ -linear neighborhood of another geodesic segment. Since $|C_\gamma^0| \geq (1 - \tilde{Q}(\eta))|C_\gamma|$, the subset $D_\gamma^0 = \{\zeta \in \mathcal{S}(\gamma, \frac{\varrho_2}{2\kappa}) : |\zeta \cap \gamma_i| \geq (1 - \frac{\varrho_1}{\varrho_2 2\kappa})|\zeta|, \text{ for some } \gamma_i \in C_\gamma^0\}$ has relative measure of at least $1 - \tilde{Q}(\eta)$, where $\tilde{Q}(\eta) \rightarrow 0$ as $\eta \rightarrow 0$.

We now tile $\mathbf{B}(\Omega)$ by $\mathbf{B}(\frac{\varrho_2}{2\kappa}\Omega)$:

$$\mathbf{B}(\Omega) = \bigsqcup_{j \in \mathbf{J}} \mathbf{B}(\Omega_j) \sqcup \Upsilon$$

where $\Omega_j = \frac{\varrho_2}{2\kappa}\Omega$. Note that the union of $\bigsqcup_j \mathcal{L}(\Omega_j)$ with the subset of $\mathcal{L}(\Omega)$ consists of elements lying in Υ is $\mathcal{L}(\Omega)$.

Set $\mathcal{L}_0 = \bigcup_{\gamma \in \mathcal{L}(\Omega)} D_\gamma^0$. The ‘favourable’ boxes are going to be those tiling boxes that have most of their geodesics belonging to \mathcal{L}_0 . That is, we set $\mathbf{J}_0 = \{j \in \mathbf{J} : |\mathcal{L}(\Omega_j) \cap (\mathcal{L}(\Omega) - \mathcal{L}_0)| \leq \tilde{q}^{1/2}|\mathcal{L}(\Omega_j)|\}$.

Then,

$$\tilde{q}^{1/2}|\mathcal{L}(\Omega_j)||\mathbf{J} - \mathbf{J}_0| \leq \sum_{j \in \mathbf{J} - \mathbf{J}_0} \tilde{q}^{1/2}|\mathcal{L}(\Omega_j)| \leq \sum_{j \in \mathbf{J}} \sum_{\zeta \in \mathcal{L}(\Omega_j)} \chi_{\mathcal{L}(\Omega) - \mathcal{L}_0} = |\mathcal{L} - \mathcal{L}_0| \leq \theta|\mathcal{L}|$$

Hence

$$|\mathbf{J} - \mathbf{J}_0| \leq \tilde{q}^{1/2} \frac{|\mathcal{L}(\Omega)|}{|\mathcal{L}(\Omega_j)|} \leq \tilde{q}^{1/2}|\mathbf{J}|O(\varrho_2)$$

□

4 Inside of a box

In this section we explore the consequences of having geodesic approximations to a large percentage of geodesic segments in a box, and extend Theorem 3.1 to the following:

Theorem 1.1 *Let G, G' be non-degenerate, unimodular, split abelian-by-abelian Lie groups, and $\phi : G \rightarrow G'$ be a (κ, C) quasi-isometry. Given $0 < \delta, \eta < \tilde{\eta} < 1$, there exist numbers $L_0, m > 1, \varrho, \hat{\eta} < 1$ depending on $\delta, \eta, \tilde{\eta}$ and κ, C with the following properties:*

If $\Omega \subset \mathbf{A}$ is a product of intervals of equal size at least mL_0 , then a tiling of $\mathbf{B}(\Omega)$ by isometric copies of $\mathbf{B}(\varrho\Omega)$

$$\mathbf{B}(\Omega) = \bigsqcup_{i \in \mathbf{I}} \mathbf{B}(\omega_i) \sqcup \Upsilon$$

contains a subset \mathbf{I}_0 of \mathbf{I} with relative measure at least $1 - \nu$ such that

- (i) *For every $i \in \mathbf{I}_0$, there is a subset $\mathcal{P}^0(\omega_i) \subset \mathcal{P}(\omega_i)$ of relative measure at least $1 - \nu'$*
- (ii) *The restriction $\phi|_{\mathcal{P}^0(\omega_i)}$ is within $\hat{\eta}\text{diam}(\mathbf{B}(\omega_i))$ Hausdorff neighborhood of a standard map $g_i \times f_i$.*

Here, ν, ν' and $\hat{\eta}$ all approach zero as $\tilde{\eta}, \delta$ go to zero.

4.1 Geometry of flats

We now observe those geometric properties of non-degenerate, unimodular, split abelian-by-abelian groups relevant to Theorem 1.1. Specifically, this subsection explores some implications of our knowledge that a large percentage of geodesics in a box admit geodesic approximations to its ϕ images.

Lemma 4.1.1. *Let G be a non-degenerate, split abelian-by-abelian group, and γ, ζ are geodesic segments in G making an angle of at least $\sin^{-1}(\tilde{\eta})$ with root kernels such that for some $\tilde{\eta} \ll \eta < 1$, $d_H(\gamma, \zeta) = \eta(|\gamma| + |\zeta|)$. Then, γ and ζ lie on a common flat for all but $\frac{\eta}{\tilde{\eta}}$ proportion of their lengths.*

Proof. If not, then there is a root α such that $\pi_\alpha(\gamma)$ and $\pi_\alpha(\zeta)$ disagrees for more than $\frac{\eta}{\tilde{\eta}}$ of their length. But this means that

$$d_H(\pi_\alpha(\gamma), \pi_\alpha(\zeta)) > \frac{\eta}{\tilde{\eta}} (|\pi_\alpha(\gamma)| + |\pi_\alpha(\zeta)|) \geq \eta (|\gamma| + |\zeta|)$$

which is a contradiction because $d_H(\gamma, \zeta) \geq d_H(\pi_\alpha(\gamma), \pi_\alpha(\zeta))$. \square

Definition 4.1.1. *Let G be a non-degenerate, split abelian-by-abelian Lie group. We define the following objects in G .*

- (i) A **2-simplex** Δ is a set of three geodesic segments that intersect pair-wisely. This includes the degenerate case of a geodesic segment and two subsegments of it. Elements of Δ are called **edges** of Δ^\dagger .
- (ii) A **filled 2-simplex** $\tilde{\Delta}$ is a set of 2-simplices $\{\Delta\} \cup \{\delta_i\}$ such that for every i , every two edges of δ_i are subsegments of two edges of Δ . The edges of Δ are called **faces** or **edges** of $\tilde{\Delta}$.

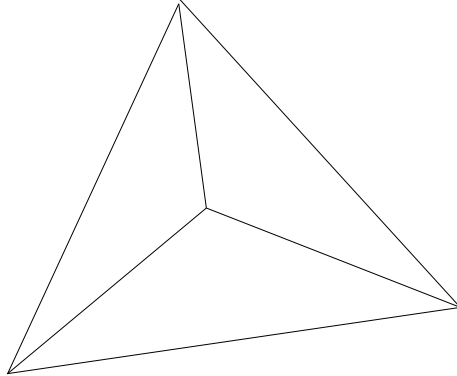


Figure 1: The big triangle together with the three little ones inside qualifies as (degenerate) 3-simplex

For $I \geq 3$, we define

[†]A 2-simplex is just a triangle. The term ‘2-simplex’ is used here only because it is more convenient to describe inductive argument later on

- (iii) A **I-simplex** Δ as a set of $I + 1$ many filled $I - 1$ -simplices such that they intersect pairwise at their $I - 2$ faces. This includes the degenerate case of a set of $I + 1$ many $I - 1$ -simplices. Elements of Δ are called **(I - 1)-faces** of Δ .
- (iv) A **filled I-simplex** $\tilde{\Delta}$ is a collection of I -simplices $\{\Delta\} \cup \{\delta_i\}$ such that for every i , I many faces of δ_i are subsets of I many faces of Δ . **Faces of $\tilde{\Delta}$** refers to faces of Δ .

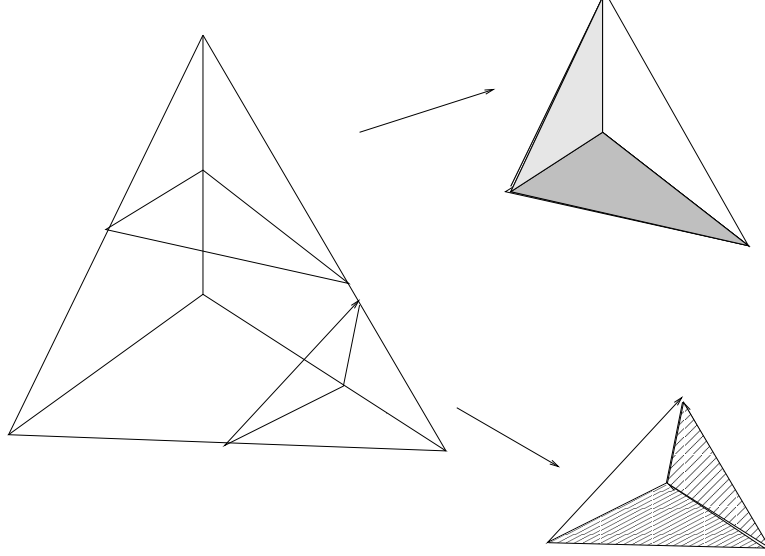


Figure 2: The big tetrahedron together with the two shaded ones qualifies as a filled 3-simplex. Note that a filled simplex cannot contain a degenerate simplex of higher dimension.

If the faces of a simplex behaves well under the quasi-isometry ϕ , that is, if ϕ images of those faces admit approximations by hyperplanes of appropriate dimensions, then we can approximate ϕ image of the simplex. This is the content of the next lemma, which deals with one instance where simplex approximations of a quasi-simplex (image of a simplex under a quasi-isometry) is possible.

Lemma 4.1.2. *Let G be a non-degenerate, split abelian-by-abelian Lie group and \mathcal{B} a family of geodesic segments such that*

- $\max\{|\zeta|, \zeta \in \mathcal{B}\} = M \ll \infty$
- *For $\zeta \in \mathcal{B}$, $\phi(\zeta)$ is within $\eta|\phi(\zeta)|$ Hausdorff neighborhood of another geodesic segment $\hat{\zeta}$. We call $\hat{\zeta}$ a geodesic approximation of $\phi(\zeta)$.*
- *For some $\tilde{\eta} \gg \eta$, the direction of any two geodesic approximation makes an angle of at most $\sin^{-1}(\eta)$, and their angles each makes an angle at least $\sin^{-1}(\tilde{\eta})$ root kernels.*
- *If $\zeta, \gamma \in \mathcal{B}$, $\gamma \subset \zeta$, and let $\hat{\zeta}, \hat{\gamma}$ be geodesic approximations of $\phi(\zeta)$ and $\phi(\gamma)$. Then there is a subsegment $\tilde{\zeta} \subset \hat{\zeta}$ such that $d_H(\tilde{\zeta}, \hat{\gamma}) \leq 2\eta|\phi(\gamma)|$.*

Then for $I \leq n$, the ϕ images of any I -simplex or filled I -simplex made out of elements of \mathcal{B} is within $O(\eta M)$ Hausdorff neighborhood of another simplex or filled simplex of the same dimension lying on a flat.

Proof. We prove the claims by induction on I , starting with a 2-simplex, then filled 2-simplex followed by 3-simplex, filled 3-simplex etc.

Base step.

2-simplex Fix three geodesic approximations for ϕ images of edges of Δ , and for each weight Ξ , look at the images of those geodesic approximations under π_Ξ . There are six possible configurations shown in figure 3 below. To specify a 2-simplex on a flat that is close to these three geodesics, it is enough to specify the root space coordinates of this flat, and this is given by the root space coordinate of the dotted line in each configuration.

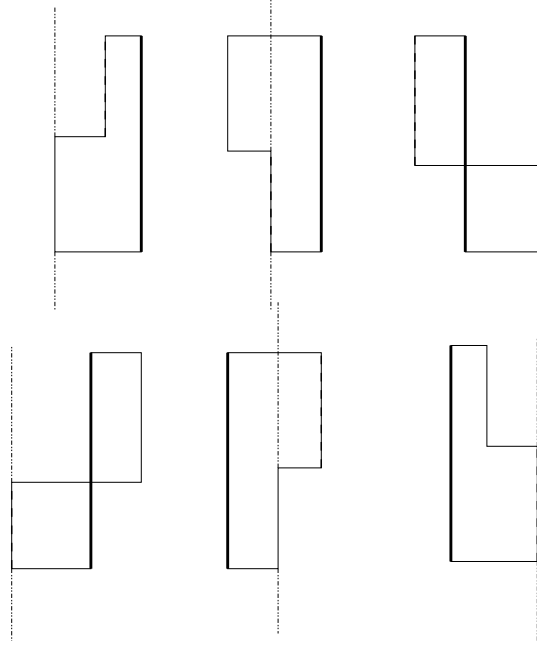


Figure 3: The six configurations in the Base step, 2-simplex case of proof to Lemma 4.1.2. The dotted line is the image of a 2-simplex close to the quasi-2-simplex.

filled 2-simplex. Let $\tilde{\Delta} = \{\Delta\} \cup \{\delta_i\}_i$, and $\hat{\Delta}$, $\hat{\delta}_i$'s denote for the 2-simplex approximation of $\phi(\Delta)$, $\phi(\delta_i)$'s, as given by **2-simplex** case above. Then for every two edges of $\hat{\delta}_i$, there are subsegments of two edges of $\hat{\Delta}$ such that each pair satisfy the hypothesis of Lemma 4.1.1. This means the flats housing $\hat{\Delta}$ and $\hat{\delta}_i$ must come together (because the conclusion of Lemma 4.1.1 says that they lie on a common flat). Since the set where two flats come together is convex, we conclude therefore that there is a 2-simplex $\tilde{\delta}_i$ lying on the flat that houses $\hat{\Delta}$ such that $d_H(\hat{\delta}_i, \tilde{\delta}_i) \leq \eta M$, and two edges of $\tilde{\delta}_i$ are subsegments of two edges of $\hat{\Delta}$. Then $\tilde{\Delta} = \{\hat{\Delta}\} \cup \{\tilde{\delta}_i\}_i$ has the desired property.

Induction step.

I -simplex Let $\Delta = \{\check{\delta}_i\}_{i=0}^I$ where each $\check{\delta}_i$ is a filled $I-1$ -simplex, and $\check{\delta}_i$ be their filled $I-1$ simplex approximations as yielded by the inductive hypothesis. Then we know for each weight Ξ , $\pi_\Xi(\check{\delta}_i)$ is a vertical geodesic segment, and for any $\check{\delta}_i, \check{\delta}_j$, $\pi_\Xi(\check{\delta}_i), \pi_\Xi(\check{\delta}_j)$ come together at some subsegment. If modulo $\frac{\eta}{\eta}$ proportion of the ends, $\pi_\Xi(\check{\delta})$'s do not lie on a common vertical geodesic segment, then the relationship between $\pi_\Xi(\check{\delta}_i), \pi_\Xi(\check{\delta}_j)$ is that of a forking Y , see Figure 4 below.

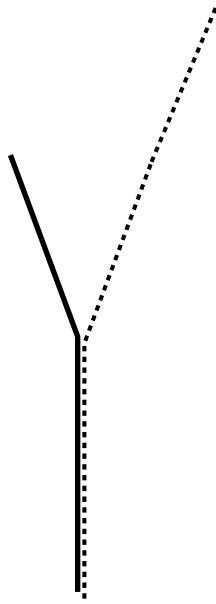


Figure 4: Inductive step, I -simplex case in the proof of Lemma 4.1.2: the solid and dotted lines represent $\pi_\Xi(\check{\delta}_i)$ and $\pi_\Xi(\check{\delta}_j)$.

But this contradicts the existence of another $\check{\delta}_k$ that shares a face with $\check{\delta}_i$ and another face with $\check{\delta}_j$. So modulo the $\frac{\eta}{\eta}$ proportion of their ends, $\pi_\Xi(\check{\delta}_i), \pi_\Xi(\check{\delta}_j)$ must lie on a common vertical geodesic. The same argument applied to every other weights means that we can translate each $\check{\delta}_i$ to $\hat{\delta}_i$ so that $\hat{\delta}_i, \hat{\delta}_j$ share a common face. The collection of all $\hat{\delta}_i$'s forms our desired $\hat{\Delta}$ I -simplex.⁵

filled I -simplex Let $\tilde{\Delta} = \{\Delta\} \cup \{\delta_i\}$ where each of Δ and δ_i is a I -simplex, and let $\hat{\Delta}$ and $\hat{\delta}_i$'s denote for I -simplex approximations of $\phi(\Delta)$ and $\phi(\delta_i)$'s as yielded above. Then for every I faces of $\hat{\delta}_i$ there are I many corresponding faces of $\hat{\Delta}$ to which they are a subset of, and this means the corresponding subsegments of edges of faces of $\hat{\Delta}$ and the edges of faces of $\hat{\delta}_i$ satisfy the hypothesis of Lemma 4.1.1, so they lie on a common flat. This means the flats housing $\hat{\Delta}$ and $\hat{\delta}_i$ respectively must come together and since the set where two flats come together is a convex set, we conclude

⁵The inductive step is not a replacement of the 2-simplex case in the Base step because here, faces intersects at filled simplex of dimension $I-2$, which has diameter compatible to that of the diameter of the I -simplices of concern, whereas in the 2-simplex case, the pair-wise intersection of edges consist of just one point for each pair, so the same forking argument wouldn't work there.

therefore that there is a I -simplex $\hat{\delta}_i$ in the flat containing $\hat{\Delta}$ such that $d_H(\hat{\delta}_i, \hat{\delta}_i) \leq \eta M$, and $\hat{\delta}$ share I of its faces with faces of $\hat{\Delta}$. Then $\tilde{\Delta} = \{\hat{\Delta}\} \cup \{\hat{\delta}_i\}$ has the desired property. \square

Definition 4.1.2. Let $\eta < 1$. A η quadrilateral $Q = \{\mathbf{T}_i\}_{i=0}^3$ in G is a set of 4 oriented geodesic segments \mathbf{T}_i 's satisfying the following:

- (i) $\exists \vec{v} \in \mathbf{A}$ for which $W_{\vec{v}}^0 = \{0\}$ such that the directions of \mathbf{T}_i 's are all parallel to \vec{v}
- (ii) $\forall i, |\mathbf{T}_i| > 2\eta \sum_{j=0}^3 |\mathbf{T}_j|$
- (iii) for all i ,
 - $d(e_i, b_{i+1}) \leq \eta(|\mathbf{T}_i| + |\mathbf{T}_{i+1}|)$,
 - $d(b_i, e_{i+1}) \geq (|\mathbf{T}_i| + |\mathbf{T}_{i+1}|)$

where b_i, e_i are the beginning and end points of \mathbf{T}_i .

We will often refer to \mathbf{T}_i 's as edges of Q , and write $\text{diam}(Q)$ for the maximum length of its edges.

Example Suppose the rank of G is 1. Let V_+, V_- denote for the two root class horocycles based at the identity element. Let $x \in V_+, y \in V_-$, and the word $xyx^{-1}y^{-1}$ represents a loop in $\mathbf{H} = V_+ \oplus V_-$. If we replace x by $t\tilde{x}t^{-1}$, and y by $t^{-1}\tilde{y}t$ for some small $\tilde{x} \in V_+$ and $\tilde{y} \in V_-$, we obtain a loop representing a quadrilateral. Note that the same construction works if G is rank 1 and non-unimodular, as long as there are two root classes.

Remark 4.1.1. The first requirement of a quadrilateral means a quadrilateral exists in the subgroup $\langle \vec{v} \rangle \rtimes \mathbf{H}$ (or a left translate of it). Since \vec{v} does not act trivially on any proper subspace, quadrilaterals exist when rank of G is 2 or higher for the same reason that they exist rank 1 spaces as illustrated by the previous example.

Lemma 4.1.3. Let $Q = \{\mathbf{T}_i\}_{i=0}^3$ be a η quadrilateral. Then the direction of \mathbf{T}_i and \mathbf{T}_{i+2} are positive multiple of each other, and that of \mathbf{T}_i and \mathbf{T}_{i+1} are negative multiple of each other.

Proof. There are 16 possibilities to the relationship among directions of all the \mathbf{T}_i 's (being positive or negative multiples of each other). One checks that only the combination stated above is allowed. An argument is given in the Appendix. \square

Let $A(t)$ be a 1-parameter matrix consisting of blocks of the form $e^{\alpha t}N(t)$ where $\alpha \neq 0$, $N(t)$ a nilpotent matrix with polynomial entries, and $\mathbb{R} \ltimes_A \mathbb{R}^m$ be a semidirect product for which $r \in \mathbb{R}$ acts on \mathbb{R}^m by linear map $A(r)$. Write an element of $\mathbb{R} \ltimes_A \mathbb{R}^m$ as (r, \mathbf{x}) , where $r \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^m$, and W^+ (resp. W^-) for the direct sum of positive (resp. negative) eigenspaces of A .

Lemma 4.1.4. In $\mathbb{R} \ltimes_A \mathbb{R}^m$, suppose for some $\eta \ll 1$, we have $r_0, r_1, r_2, r_3 > 0$, $u_0, u_2 \in W^+$, $u_1, u_3 \in W^-$ satisfying

- $d(u_j, e) \leq \eta(r_j + r_{j+1}), \forall j$
- $r_j > 2\eta \sum_{\iota=0}^3 r_{\iota}, \forall j$

- The word $(r_0, 0)u_0(-r_1, 0)u_1(r_2, 0)u_2(-r_3, 0)u_3$ is trivial.

Then $|r_i - r_{i+1}| \leq d(e, u_{i+1}) + d(e, u_{i+3})$. In particular this implies that the sizes of r_i 's are equal up to an error of at most $\eta \sum_{i=0}^3 r_i$.

Proof. See Appendix □

Lemma 4.1.5. Let $Q = \{\mathbf{T}_i\}_{i=0}^3$ be a η quadrilateral. Then

$$(i) \quad |\mathbf{T}_i| - |\mathbf{T}_j| \leq \eta \left(\sum_{i=0}^3 |\mathbf{T}_i| \right), \quad \forall i, j$$

$$(ii) \quad \forall i, \quad |\pi_{\vec{v}} \circ \Pi_{\vec{v}}(e_i) - \pi_{\vec{v}} \circ \Pi_{\vec{v}}(b_{i-1})| \leq d(e_i, b_{i+1}) + d(e_{i+2}, b_{i+3})$$

$$(iii) \quad \{\Pi_{\vec{v}}(b_i), \Pi_{\vec{v}}(e_{i+1}), \Pi_{\vec{v}}(b_{i+2}), \Pi_{\vec{v}}(e_{i+3})\} \text{ are within } \eta \left(\sum_{i=0}^3 |\mathbf{T}_i| \right) \text{ neighborhood of a coset of } W_{\vec{v}}^+ \\ \text{(or } W_{\vec{v}}^- \text{) if } i = 0 \pmod{2}, \text{ and of a coset of } W_{\vec{v}}^- \text{ (or } W_{\vec{v}}^+ \text{) otherwise.}$$

Proof. Modifying \mathbf{T}_i 's by an amount of at most $\eta \sum_j |\mathbf{T}_j|$, we can assume $\pi_A(e_i) = \pi_A(b_{i+1})$ for all i . Furthermore, the divergent assumption between b_i and e_{i+1} means that $e_i^{-1}(b_{i+1}) \in W_{\vec{v}}^+$ (resp. $W_{\vec{v}}^-$) if the direction of \mathbf{T}_i is positive (resp. negative) multiples of \vec{v} . The result now follows from Lemma 4.1.4. □

A schematic illustration for a quadrilateral with the correct orientation and lengths for its edges is given in Figure 5 below.

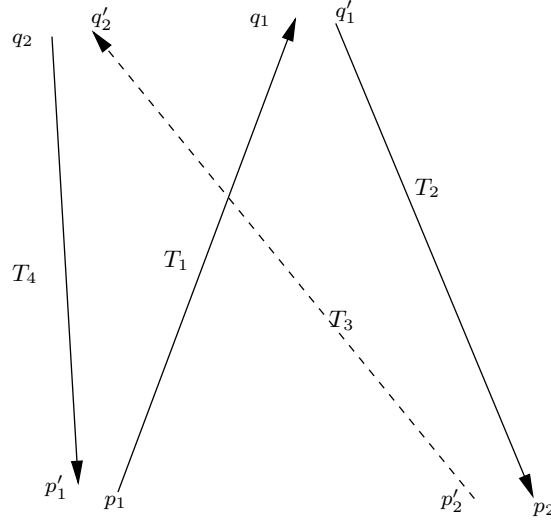


Figure 5: A schematic illustration of a quadrilateral

Lemma 4.1.6. *Let $Q = \{\gamma_j\}_{j=0}^3$ be a 0-quadrilateral in G , such that each γ_j is properly contained in a geodesic segment $\tilde{\gamma}_j$, whose ϕ image is within $\eta|\tilde{\gamma}|$ neighborhood of another geodesic segment whose direction is parallel to $\vec{v}_j \in \mathbf{A}$ with $W_{\vec{v}_j}^0 = \{0\}$. Suppose further that each $|\gamma_j| > 2\eta \sum_{\iota} |\tilde{\gamma}_{\iota}|$. Then, there is a $\hat{\eta} (= \max\{\eta \frac{|\tilde{\gamma}_j|}{|\gamma_j|}\})$ -quadrilateral \hat{Q} satisfying $d_H(\phi(Q), \hat{Q}) \leq \hat{\eta} \text{diam}(Q)$.*

Proof. For each j , let \tilde{T}_j be an geodesic approximation of $\phi(\tilde{\gamma}_j)$. Since Q is a 0-quadrilateral, $\tilde{\gamma}_j \cap \tilde{\gamma}_{j+1}$ is a geodesic segment with positive length, therefore $\angle(\vec{v}_j, \vec{v}_{j+1}) \leq \sin^{-1}(\eta)$, and $d(\tilde{T}_j, \tilde{T}_{j+1}) \leq \eta(|\tilde{T}_j| + |\tilde{T}_{j+1}|)$. By moving each \tilde{T}_j by an amount at most $\sum_{\iota} |\tilde{T}_{\iota}|$, we can assume the directions of \tilde{T}_j 's are all parallel to some \vec{v} with $W_{\vec{v}}^0 = \{0\}$, and $\tilde{T}_j \cap \tilde{T}_{j+1}$ is a geodesic segment of positive length. Let $\mathbf{T}_j \subset \tilde{T}_j$ be the subsegment closest to $\phi(\gamma_j)$. Then $\hat{Q} = \{\mathbf{T}_j\}$ is a $\hat{\eta}$ -quadrilateral. \square

4.2 Averaging

In this subsection, we put together some of the observations in the last two subsections to show that if a large percentage of geodesic segments in a box admit geodesic approximations to their ϕ images, then for $i \geq 2$, a large percentage of i -hyperplanes in the box also admit i -hyperplane approximations to their ϕ images. In particular, there is a large subset of flats in the box whose ϕ images are close flats.

The following averaging lemma that will be used repeatedly for the remaining of this section.

Lemma 4.2.1. *Let $(A, \mu_{\alpha}), (B, \mu_{\beta})$ be two finite measure space, and \sim is a symmetric relation between them. For $a \in A$, write $B_a = \{b \in B, b \sim a\}$ as the subset of B consisted of elements related to a , and A_b , for $b \in B$, as the subset of elements of A related to b . Suppose $\frac{\mu_{\beta}(B_a)}{\mu_{\beta}(B_{a'})} \leq M_A$ for any $a, a' \in A$, and $\frac{\mu_{\alpha}(A_b)}{\mu_{\alpha}(A_{b'})} \leq M_B$ for any $b, b' \in B$.*

If for some $s \leq \frac{1}{M_A M_B}$, $A_s \subset A$ with $\mu_{\alpha}(A_s) \leq s\mu_{\alpha}(A)$, then the subset $B^{s,t} = \{b \in B : \mu_{\alpha}(A_b \cap A_s) \geq t\mu_{\alpha}(A_b)\}$, satisfies $\mu_{\beta}(B^{s,t}) \leq \frac{s}{t} M_A M_B \mu_{\beta}(B)$.

Proof. See Appendix. \square

Remark 4.2.1. *Lemma 4.2.1 will often be used to show that for subset $A_0 \subset A$ of relative large measure, the subset of B consisting of elements $b \in B$ such that the measure of $A_b \cap A_0$ is large relative to that of A_b , is large.*

Lemma 4.2.2. *Let μ be a $\mathbb{O}(k+1)$ -invariant measure on $\mathbb{O}(k+1)/\mathbb{O}(k)$. Let $\{e_{\iota}\}_{\iota=1}^{k+1}$ be an orthonormal basis of \mathbb{R}^{k+1} and H_j be linear span of $\{e_{\iota}\}_{\iota \neq j}$, M_k as the common value of $d(H_j, H_{j'})$ in $\mathbb{O}(k+1)/\mathbb{O}(k)$. Suppose for some $v \ll 1$, Ω is a subset with $\mu(\Omega) \geq (1-v)\mu(\mathbb{O}(k+1)/\mathbb{O}(k))$. Then we can find $k+1$ points $x_i \in \Omega$ such that $d(x_i, x_j) \geq M_k - W(v)$, where $W(v) \rightarrow 0$ as $v \rightarrow 0$.*

Proof. Equip $\mathbb{O}(k+1)/\mathbb{O}(k) \times \mathbb{O}(k+1)/\mathbb{O}(k)$ with the product measure $\mu \times \mu$. Then the relative measure of $\Omega \times \Omega$ is at least $(1-v)^2$. If the claim was not true, then Ω is contained in a ball of radius M_k , and this would create a contradiction to the measure of $\Omega \times \Omega$ when v is sufficiently small. \square

Lemma 4.2.3. *Let $G = \mathbf{H} \rtimes \mathbf{A}$, $G' = \mathbf{H}' \rtimes \mathbf{A}'$ be non-degenerate, unimodular, split abelian-by-abelian Lie groups, and $\phi : G \rightarrow G'$ be a (κ, C) quasi-isometry. Let $\Omega \subset \mathbf{A}$ be a product of intervals of equal size. Inside of the box $\mathbf{B}(\Omega) \subset G$, suppose for some $\eta < 1$ there is a subset $\mathcal{L}^0 \subset \mathcal{L}(\Omega)[m]$,*

where $m \rightarrow \infty$ as $\eta \rightarrow 0$, and $|\mathcal{L}^0| \geq (1 - F_1)|\mathcal{L}(\Omega)[m]|$, where F_1 is a function of η and approaches zero as $\eta \rightarrow 0$, such that for every $l \in \mathcal{L}^0$,

(i) $\phi(l)$ is within $\eta|l|$ Hausdorff neighborhood of a geodesic segment that makes an angle at least $\sin^{-1}(\tilde{\eta})$ with root kernel directions. Here $\tilde{\eta}$ depends on η and approaches zero as $\eta \rightarrow 0$.

(ii) For each $l \in \mathcal{L}^0$, the proportion of η uniform points is at least $1 - F_1$.

Then, for $i = 2, 3, \dots, \dim(\mathbf{A})$, there are subsets $\mathcal{L}_i^0 \subset \mathcal{L}_i(\Omega)[m]$ and functions F_i , together with a subset $\mathcal{P}^0 \subset \mathcal{P}(\Omega)$ and a function F_0 that satisfy the the following properties.

(i) F_i 's and F_0 are functions of η and approach zero as $\eta \rightarrow 0$.

(ii) If $S \in \mathcal{L}_i^0$, then $\phi(S)$ is within $\eta \text{diam}(\mathbf{B}(\Omega))$ Hausdorff neighborhood of a i -dimensional hyperplane.

(iii) For every $p \in \mathcal{P}^0$,

$$|L(p) \cap \mathcal{L}(\Omega)[m] \cap \mathcal{L}^0| \geq (1 - F_1)^2 |L(p) \cap \mathcal{L}(\Omega)[m]|$$

and $i = 2, 3, \dots$,

$$|L_i(p) \cap \mathcal{L}_i(\Omega)[m] \cap \mathcal{L}| \geq (1 - F_i)^2 |L_i(p) \cap \mathcal{L}_i(\Omega)[m]|$$

(iv) The relative measure of \mathcal{L}_i^0 and \mathcal{P}^0 in $\mathcal{L}_i(\Omega)[m]$ and $\mathcal{P}(\Omega)$ are at least $1 - F_i$ and $1 - F_0$ respectively.

Proof. More precisely, we prove the following claims:

For $i = 2, 3, \dots, \dim(\mathbf{A})$, there are subsets $\mathcal{L}_i^0 \subset \mathcal{L}(\Omega)[m]$, $\mathcal{P}_i \subset \hat{\mathcal{P}}_i$ of $\mathcal{P}(\Omega)$, all of relative large measure such that

- a. elements of \mathcal{L}_{i-1}^0 admit $i - 1$ hyperplane approximations. \mathcal{L}_1^0 is defined to be \mathcal{L}^0 .
- b. if $S \in \mathcal{L}_i^0$, $p \in \mathcal{P}_i$, $p \in S$, then a large proportion of elements in $L_{i-1}(p) \cap L_{i-1}(S)$ lies in \mathcal{L}_{i-1}^0 . Here, 'large' means closer to 1 as $\eta \rightarrow 0$.
- c. Elements of \mathcal{L}_i^0 admit i -hyperplane approximations.

The proof proceeds in two steps. First, we construct $\mathcal{L}_i^0(\Omega)$ and subsets $\mathcal{P}_i \subset \mathcal{P}(\Omega)$ by induction. Then we use \mathcal{P}_i to show that elements of \mathcal{L}_i^0 satisfies the desired properties. The set \mathcal{P}^0 will be the intersection of those \mathcal{P}_i from the first step.

We start with the base case when $i = 2$.

The incidence relation between $\mathcal{L}(\Omega)[m]$ and $\mathcal{L}_2(\Omega)[m]$ is symmetrical. Therefore, by Lemma 4.2.1 we can choose $s_2(\eta) \ll 1$ appropriately so that the set

$$\mathcal{L}_2^0 = \{S \in \mathcal{L}_2(\Omega)[m] : |L_1(S) \cap \mathcal{L}(\Omega)[m] \cap \mathcal{L}^0| \geq (1 - s_2) |L_1(S) \cap \mathcal{L}(\Omega)[m]|\}$$

satisfies

$$|\mathcal{L}_2^0| \geq (1 - F_1^{1/2}) |\mathcal{L}(\Omega)[m]|$$

Fix a $S \in \mathcal{L}_2^0$. Let $P(S)^{bad} \subset P(S)$ consisting of those points p such that p fails to be uniform with respect to at least s_b proportion of elements in $L(p) \cap L(S) \cap \mathcal{L}(\Omega)[m]$. Note that this means if ζ is not an element of $L(p) \cap L(S) \cap \mathcal{L}^0$, then p is not uniform with respect to ζ . We obtain a bound on the relative size of $P(S)^{bad}$ as follows.

Let χ be the characteristic function of the subset of $\{(p, \zeta) : p \in P(S), \zeta \in L(S) \cap \mathcal{L}(\Omega)[m], p \in \zeta\}$ consisting of pairs (p, ζ) such that either $\zeta \notin \mathcal{L}^0$ or $\zeta \in \mathcal{L}^0$ but p fails to be a uniform point on it.

Then, starting from

$$\sum_{x \in P(S)} \sum_{\gamma \in L(x) \cap L(S) \cap \mathcal{L}(\Omega)[m]} \chi = \sum_{\gamma \in L(S) \cap \mathcal{L}(\Omega)[m]} \sum_{p \in P(\gamma)} \chi$$

we have

$$\begin{aligned} \sum_{x \in P(S)} \sum_{\gamma \in L(x) \cap L(S) \cap \mathcal{L}(\Omega)[m]} \chi &\geq \sum_{x \in P(S)^{bad}} s_b |L(x) \cap L(S) \cap \mathcal{L}(\Omega)[m]| \\ &\geq s_b |P(S)^{bad}| |L(x) \cap L(S) \cap \mathcal{L}(\Omega)[m]|_{\min, x \in S} \end{aligned}$$

and

$$\begin{aligned} \sum_{\gamma \in L(S) \cap \mathcal{L}(\Omega)[m]} \sum_{p \in \gamma} \chi &= \sum_{\gamma \in L(S) \cap (\mathcal{L}(\Omega)[m] - \mathcal{L}^0)} \sum_{p \in \gamma} \chi + \sum_{\gamma \in L(S) \cap \mathcal{L}^0} \sum_{p \in \gamma} \chi \\ &\leq \sum_{\zeta \in L(S) \cap \mathcal{L}(\Omega)[m]} F_1 |P(\zeta)| + \sum_{\gamma \in L(S) \cap \mathcal{L}^0} F_1 |P(\gamma)| \\ &\leq \sum_{\zeta \in L(S) \cap \mathcal{L}(\Omega)[m]} F_1 |P(\zeta)| + \sum_{\gamma \in L(S) \cap \mathcal{L}(\Omega)[m]} F_1 |P(\gamma)| \\ &\leq 2F_1 \sum_{\zeta \in L(S) \cap \mathcal{L}(\Omega)[m]} |P(\zeta)| \\ &\leq 2F_1 |L(S) \cap \mathcal{L}(\Omega)[m]| |P(\zeta) \cap P(S)|_{\max, \zeta \in \mathcal{L}(\Omega)[m]} \end{aligned}$$

which yields

$$|P(S)^{bad}| \leq \frac{2F_1}{s} k |P(S)|$$

where k depends only on G . By choosing $s_b = 2F_1^{1/2}$, we have the measure of $P(S)^{bad}$ is at least $1 - F_1^{1/2}$ times that of $P(S)$.

We now apply Lemma 4.2.1 to $P(S)$, $L(S) \cap \mathcal{L}(\Omega)[m]$, and $P(S)^{bad}$ to conclude that for some $\nu_2 \ll 1$, the subset

$$L(S)^{bad} = \{\zeta \in L(S) \cap \mathcal{L}(\Omega)[m] : |P(\zeta) \cap P(S)^{bad}| \geq \nu_2 |P(\zeta)|\}$$

satisfies

$$|L(S)^{bad}| \leq \nu_2' |L(S)|$$

for some $\nu_2' \ll 1$.

Now apply Lemma 4.2.1 again to $P(S)$, $L(S) \cap \mathcal{L}(\Omega)[m]$, and $L(S)^{good} = (L(S) - L(S)^{bad}) \cap \mathcal{L}^0$ to conclude that for some $\hat{\nu}_2 \ll 1$, the subset

$$P(S)^w = \{p \in P(S) : |L(p) \cap \mathcal{L}(\Omega)[m] \cap L(S)^{good}| \leq (1 - \hat{\nu}_2) |L(p) \cap \mathcal{L}(\Omega)[m]| \}$$

satisfies

$$|P(S)^w| \leq \tilde{\nu}_2 |P(S)|$$

for some $\tilde{\nu}_2 \ll 1$. Now set $P(S)^0$ as $P(S) - P(S)^{bad} - P(S)^w$, and let \mathcal{P}_2 as the union of $P(S)^0$ as S ranges over \mathcal{L}_2^0 .

Now run the same argument for $i = 3$, replacing ‘uniform points’ of an element of \mathcal{L}^0 by $P(S)^0$, where $S \in \mathcal{L}_2^0$. Repeat this procedure inductively, to arrive at subsets \mathcal{L}_i^0 , and $P(S)^0 \subset P(S)$ for every $S \in \mathcal{L}_i^0$, all of relative large measure.

For a $S \in \mathcal{L}_{i+1}^0$, we now show that $\phi(P(S)^0)$ is close to a $i + 1$ -dimensional hyperplane. We will do this by induction. The hypothesis furnishes the base step.

Take $p, q \in P^{good}(S)$. By construction, for $\hat{\nu}_i, \mu_i \ll 1$, the ϕ images of at least $1 - \hat{\nu}_i$ proportion of elements in $L_i(p)$ and $L_i(q)$ have the properties that 1) spend at least $1 - \mu_i$ proportion of their area/measure in $P^0(S)$, and 2) belong to \mathcal{L}_i^0 , so admit i -hyperplane approximations by inductive hypothesis.

There are two cases to consider.

Case I. At least one of p, q is at least $\eta \text{diam}(\mathbf{B}(\Omega))$ away from $\partial \mathbf{B}(\Omega)$, in which case we can do one of the followings: (see also Figure 6 below.)

- find i many points $r_\iota \in P^0(S)$, $\iota = 1, 2, \dots, i$ and $Q_p \in L_i(p) \cap L_i(S) \cap \mathcal{L}_i^0$, $Q_{r_\iota} \in L_i(r_\iota) \cap L_i(S) \cap \mathcal{L}_i^0$ such that they intersect to form a $i + 1$ -simplex Δ with p, q and r_ι ’s lying on its faces.
- or pick an element $Q_p \in L_i(p) \cap L_i(S) \cap \mathcal{L}_i^0$. Since Q_p has codimension 1 in S , most elements of $L_i(q) \cap L_i(S) \cap \mathcal{L}_i^0$ will intersect it and we can find i many elements $Q_{q,\iota} \in L_i(q) \cap L_i(S) \cap \mathcal{L}_i^0$, $\iota = 1, 2, \dots, i$ such that they intersect Q_p to make a $i + 1$ -simplex Δ with q being one of its vertices and p lying on the face opposite to q .

We now apply Lemma 4.1.2 to conclude that the ϕ images of Δ are within $\eta \text{diam}(\mathbf{B}(\Omega))$ neighborhood of another $i + 1$ -simplex on a $i + 1$ -dimensional hyperplane.

Case II. Both p and q within $\eta \text{diam}(\mathbf{B}(\Omega))$ of $\partial \mathbf{B}(\Omega)$.

In this case we make a $i + 1$ -simplex with p as one of its vertex as follows. (see also Figure 7 below) Apply Lemma 4.2.2 to subsets $L_i(p) \cap L_i(S) \cap \mathcal{L}_i^0$, which allows us to pick out $i + 1$ elements $Q_{p,\iota} \in L_i(p) \cap L_i(S) \cap \mathcal{L}_i^0$ that are almost equally spaced apart (up to an error of $W(\eta)$ by Lemma 4.2.2).

Since each $Q_{p,\iota}$ spends at least $1 - \mu_i$ proportion of its measure in the set $P^0(S)$, we can certainly find $x \in Q_{p,1} \cap P^0(S)$. Furthermore we can assume x is at most $O(\eta \text{diam}(\mathbf{B}(\Omega)))$ away from $\partial \mathbf{B}(\Omega)$.

The subset of $L_i(x)$ that intersect all of $Q_{p,\iota}$ ’s, for $\iota = 1, 2, \dots, i$ has large positive measure because elements of $L_i(x)$ has codimension 1 in S . So we can find $Q_x \in L_i(x) \cap L_i(S)$ that intersects all of $Q_{p,\iota}$ ’s, thus making a $i + 1$ -simplex Δ . By choice, faces of Δ : $Q_{p,1}, Q_{p,2} \dots Q_{p,i}, Q_x$, when considered as points in $\mathbb{O}(i + 1)/\mathbb{O}(i)$, have pair-wise distance at least $M_i - W(\eta)$, which means the volume of the set bounded by Δ is at least $\frac{1}{2^i}$ proportion of the volume of S .

Let $z \in P^0(S)$ be a point that lies in the interior of the set bounded by Δ . Then most elements of $L_i(z) \cap L_i(S) \cap \mathcal{L}_i^0$ are going to intersect $i + 1$ faces of Δ thus making a smaller $i + 1$ -simplices, i

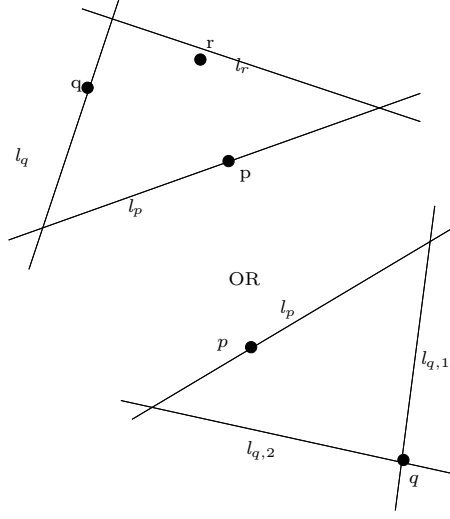


Figure 6: Case I: two ways of making a simplex when p, q are far from the boundary of the box.

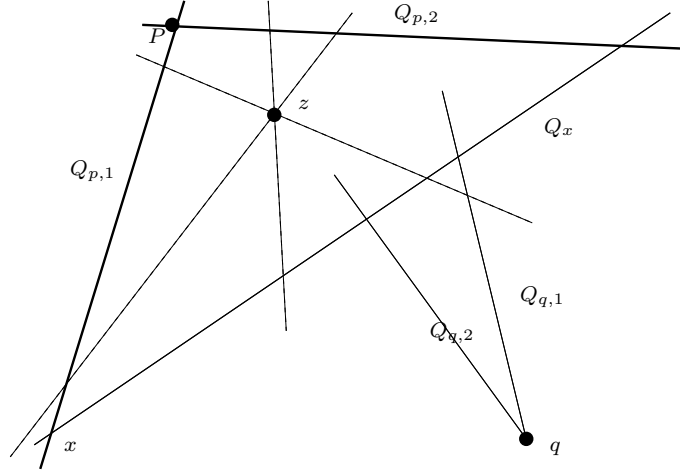


Figure 7: Case II: when either p or q is too close to the boundary of the box, we first make a filled simplex using p and look at the intersection between i -hyperplanes in $L_i(q)$ that intersect this filled simplex.

of its faces are subsets of faces of Δ . We construct such $i + 1$ -simplices δ_i for all points in $P^0(S)$, and the collection of them together with Δ gives us $\tilde{\Delta} = \{\Delta\} \cup \{\delta_i\}$ a filled $i + 1$ -simplex.

By Lemma 4.1.2, ϕ image $\tilde{\Delta}$ is within $\eta \text{diam}(\mathbf{B}(\Omega))$ Hausdorff neighborhood from another filled $i + 1$ -simplex $\check{\Delta}$ on $i + 1$ hyperplane. We are done if in addition, $q \in \tilde{\Delta}$. If not, then we can find two elements $Q_{q,1}, Q_{q,2} \in L_i(q) \cap L_i(S) \cap \mathcal{L}_i^0$ such that they both have no empty intersection with $\tilde{\Delta}$, because the area of the set bounded by Δ to that of S is at least $1/2^i$.

Let $\hat{Q}_{q,1}, \hat{Q}_{q,2}$ be i -hyperplane approximations to $\phi(Q_{q,1})$ and $\phi(Q_{q,2})$. Then for any root Ξ , π_Ξ images of $\hat{Q}_{q,1}$ and $\hat{Q}_{q,2}$ on the ends away from $\pi_\Xi(\phi(q))$ lie on a common vertical geodesic segment because they both intersect $\tilde{\Delta}$ which lie on a $i + 1$ -hyperplane, and on the $\pi_\Xi(\phi(q))$ end, the vertical geodesic segment containing them come together because both $Q_{q,i}$'s contain q . Since any two geodesic segments in a hyperbolic space come together at most one end, this means $\pi_\Xi(\hat{Q}_{q,1})$ and $\pi_\Xi(\hat{Q}_{q,2})$ lie on a common vertical geodesic segment. As Ξ ranges over all roots, this means that $\hat{Q}_{q,1}$ and $\hat{Q}_{q,2}$ lie on the same flat as $\tilde{\Delta}$. Lastly, as $\tilde{\Delta}$ lie on a $i + 1$ -hyperplane and each of $\hat{Q}_{q,i}$'s is a i -hyperplane, this gives us $\tilde{\Delta} \cup \{\hat{Q}_{q,i}\}$ lie on a common $i + 1$ -hyperplane within a flat. \square

4.3 Proof of Theorem 1.1

Proof. of Theorem 1.1

Apply Theorem 3.1 to $\mathbf{B}(\Omega)$. Take a $\mathbf{B}(\Omega_j)$, $j \in \mathbf{J}_0$ and apply Lemma 4.2.3 to obtain subsets $\mathcal{L}^0 \subset \mathcal{L}(\Omega_j)[m]$, $\mathcal{L}_\iota^0 \subset \mathcal{L}_\iota(\Omega_j)[m]$ for $\iota = 2, 3, \dots, \text{rank}(G)$, and $\mathcal{P}^0 \subset \mathcal{P}(\Omega_j)$, all with relative measures approaching 1 as η, δ approach zero, such that if $\zeta \in \mathcal{L}^0$, then $\phi(\zeta)$ is within $2\kappa\eta|\zeta|$ Hausdorff neighborhood of a geodesic segment that makes an angle at most $\sin^{-1}(\tilde{\eta})$ with root angles. While when S is an element of \mathcal{L}_ι^0 for some $\iota = 2, 3, \dots, \text{rank}(G)$, ϕ images of the subset of \mathcal{P}^0 lying in S are within $\eta \text{diam}(\mathbf{B}(\Omega_j))$ of a hyperplane of appropriate dimension. This means that the restriction of $\phi|_{\mathbf{B}(\Omega_j)}$ to the subset \mathcal{P}^0 sends left cosets of \mathbf{A} to left cosets of \mathbf{A}' up to an error of $\eta \text{diam}(\mathbf{B}(\Omega_j))$.

From now on we drop the subscript j . Let $\mu = (\tilde{\eta})^{1/2}$ and tile $\mathbf{B}(\Omega)$ by $\mathbf{B}(\mu\Omega)$:

$$\mathbf{B}(\Omega) = \bigsqcup_{i \in \mathbf{I}} \mathbf{B}(\omega_i) \cup \Upsilon$$

By Lemma 2.2.1, we can assume each of the tiles $\mathbf{B}(\omega_i)$ is at least $\mu \text{diam}(\mathbf{B}(\Omega))$ away from the boundary of $\mathbf{B}(\Omega)$, and the measure of Υ is at most $O(\tilde{\eta})$ times that of $\mathbf{B}(\Omega)$.

By Chebyshev inequality and Lemma 4.2.1 we can obtain a subset $\mathbf{I}_0 \subset \mathbf{I}$ with $|\mathbf{I}_0| \geq (1 - \varsigma)|\mathbf{I}|$ such that for every $i \in \mathbf{I}_0$, there are subsets $\mathcal{L}^0(\omega_i)$, $\mathcal{L}_{\text{rank}(G)}^0(\omega_i)$ and $\mathcal{P}^0(\omega_i)$ of $\mathcal{L}(\omega_i)$, $\mathcal{L}_{\text{rank}(G)}(\omega_i)$, and $\mathcal{P}(\omega_i)$, all of relative measure at least $1 - v$ whose elements are restriction of \mathcal{L}^0 , $\mathcal{L}_{\text{rank}(G)}^0$ and \mathcal{P}^0 to $\mathbf{B}(\omega_i)$. Here, ς and v both go to zero as $\tilde{\eta} \rightarrow 0$.

Take a $\mathbf{B}(\omega_i)$, $i \in \mathbf{I}_0$. Then the restriction of $\phi|_{\mathbf{B}(\omega_i)}$ to $\mathcal{P}^0(\omega_i)$ sends flats to within $\frac{\eta}{\mu} \text{diam}(\mathbf{B}(\omega_i))$ Hausdorff distance of a flat. Note that $\eta < \tilde{\eta} < 1$, so $\frac{\eta}{\mu} \ll 1$ and approaches zero when $\tilde{\eta} \rightarrow 0$. Since two flats come together at a convex set whose boundary is a union of hyperplanes parallel to root kernels.

To obtain a product structure on \mathcal{P}^0 , we proceed to show that $\phi|_f$ and $\phi|_{f'}$ for $f, f' \in \mathcal{L}_{\text{rank}(G)}^0$ are identical up to a translational error of $\eta \text{diam}(\mathbf{B}(\Omega_j))$. In the process of doing so, we will also show that left cosets of \mathbf{H} are sent to left cosets of \mathbf{H}' up to an error of the same order.

First we show that the claim is true for two flats $f, f' \in \mathcal{L}_{rank(G)}^0(\omega_i)$ that are at least $8\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$ units apart and contains points $p \in f \cap \mathcal{P}^0(\omega_i)$, $p' \in f' \cap \mathcal{P}^0(\omega_i)$ such that p, p' lie on a common root class horocycle.

Since $p, p' \in \mathcal{P}^0(\omega_i) \subset \mathcal{P}^0$, we can find geodesic segments $l_{p,1}, l_{p,2} \in \mathcal{L}^0(\Omega)$ containing p , $l_{q,1}, l_{q,2} \in \mathcal{L}^0(\Omega)$ containing q such that for some subsegments $\hat{l}_{*,\iota} \subset l_{*,\iota}$, $*$ = p, q , $\iota = 1, 2$, $Q = \{\hat{l}_{p,\iota}, \hat{l}_{q,\iota}\}_{\iota=1,2}$ is a 0 quadrilateral.

As $d(p, p') \geq 8\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$, by Lemma 4.1.6, there is a η quadrilateral \hat{Q} within $\eta diam(\mathbf{B}(\Omega_j))$ (i.e. $\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$) Hausdorff distance away from $\phi(Q)$. Applying Lemma 4.1.5 to \hat{Q} , we see that $\phi(p)$ and $\phi(p')$ are within $\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$ neighborhood of a left translate of $W_{\vec{v}}^+$ or $W_{\vec{v}}^-$ where \vec{v} is the direction of edges of \hat{Q} . Since $p, p' \in \mathcal{P}^0(\omega_i)$, we can build quadrilaterals Q_1, Q_2, \dots, Q_k for $k \leq n+2$, the edges of each are elements of $\mathcal{L}^0(\Omega)$ such that their respective approximating quadrilaterals $\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_k$, with edge directions $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ satisfies $\cap_{\iota=1}^k W_{\vec{v}_\iota}^{\sigma(\iota)}$ with $\sigma(\iota) \in \{+, -\}$, is $V_{[\alpha]}$ for some root class $[\alpha]$. Argue as before, we see that $\phi(p)$ and $\phi(q)$ lie within $\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$ Hausdorff neighborhood of a translate $W^{\sigma(\iota)}$ for $\iota = 1, 2, \dots, k$, therefore $\phi(p)$ and $\phi(q)$ lie within $\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$ Hausdorff neighborhood of a translate of $V_{[\alpha]}$.

By using more quadrilaterals, the argument above also shows that $\phi|_{f \cap \mathcal{P}^0(\omega_i)}$ are the same as $\phi|_{f' \cap \mathcal{P}^0(\omega_i)}$ up to an error of $\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$.

In general, for two arbitrary points $p, p' \in \mathcal{P}_i^0$ in the same left coset of \mathbf{H} , we can find at most $|\Delta|$ number of points $p_0 = p, p_1, p_2, \dots, p_l = p'$, such that each pair of successive points lie on a common root class horocycle. The quadrilateral argument above then shows that $\phi(p)$ are $\phi(q)$ within $|\Delta|\frac{\eta}{\mu}diam(\mathbf{B}(\omega_i))$ Hausdorff neighborhood of a translate of \mathbf{H}' . \square

References

- [A] L. Auslander. An exposition of the structure of solvmanifolds, Bull. Amer. Math. Soc (1973), 227-285
- [BH] M. Bridson, A. Haefliger. Metric Spaces of non-positive curvature. Springer-Verlag Berlin Heidelberg 1999
- [C] Y. de Cornulier. Dimension of asymptotic cones of Lie groups.
- [D] T. Dymarz. Large scale geometry of certain solvable groups. Preprint.
- [EFW0] A. Eskin, D. Fisher, K. Whyte. Quasi-isometries and rigidity of solvable groups. Preprint. Pur. Appl. Math. Q.
- [EFW1] A. Eskin, D. Fisher, K. Whyte. Coarse differentiation of quasi-isometries I: spaces not quasi-isometric to Cayley graphs.
- [EFW2] A. Eskin, D. Fisher, K. Whyte. Coarse differentiation of quasi-isometries II: Rigidity for Sol and Lamplighter groups, preprint
- [K] A.W.Knapp. Lie groups beyond an introduction. Birkhauser.
- [O] D. Osin. Exponential radicals of solvable Lie groups, J. Algebra 248 (2002), 790-805.

[P] I. Peng. Coarse differentiation and quasi-isometries of a class of solvable Lie groups II. Preprint.

Appendix

Proof. of Lemma 3.1.1 We will use the notations from equation (2). Write $p = (x, t)$, $q = (x', t')$. By assumption, $|t - t'| \leq s$. If $U(|x - x'|) \leq \min\{t, t'\}$, then assume $t \geq t'$

$$d((x, t), (x', t')) \leq d((x, t), (x', t)) + d((x', t), (x', t')) \leq 2(t - t') + 1 \leq 3s$$

and we are done.

Now suppose $U|x - x'| \geq t, t'$, but $U|x - x'| \leq 4s$, then

$$\begin{aligned} d((x, t), (x', t')) &\leq d((x, t), (x, U(|x - x'|))) + d((x, U(|x - x'|)), (x', U(|x - x'|))) \\ &\quad + d((x', U(|x - x'|)), (x', t')) \leq 2U(|x - x'|) - (t + t') + 1 \\ &\leq 8s + 1 \leq 12\kappa s \end{aligned}$$

and we are done.

Finally suppose $U(|x - x'|) \geq t, t'$, and $U(|x - x'|) \geq 4s$. Since η is continuous, we can find $i_0 \leq i_1 \leq i_2 \leq i_3 \dots i_n \in [a, b]$ and therefore points $\{p_j = \eta(i_j)\}_{j=1}^n$ such that $p = (x, t) = \eta(i_0) = p_0$, $p_n = \eta(i_n) = q = (x', t')$, and $U(|x_j - x_{j+1}|) = 4s$, for all j except maybe the last one, where $U(|x_{n-1} - x_n|) \leq 4s$.

Then by equation (2)

$$\frac{\sum_{j=0}^{n-1} (U(|x_j - x_{j+1}|) - (t_j + t_{j+1}))}{(U(|x_0 - x_n|) - (t_0 + t_n))} \leq \frac{\sum_{j=0}^{n-1} d(p_j, p_{j+1})}{d(p_0, p_n)} \leq 2\kappa$$

Simplifying using equation (3) yields

$$\frac{(n-1)2s}{2 \ln(ne^{4s})} \leq 2\kappa$$

which means

$$\begin{aligned} (n-1)s &\leq 2\kappa (\ln(n) + 4s) \\ ns - 2\kappa \ln(n) &\leq s + 8\kappa s \\ \frac{1}{2}ns &\leq ns - 2s \ln(n) \leq ns - 2\kappa \ln(n) \leq s + 8\kappa s \leq 9\kappa s \\ n &\leq 20\kappa \end{aligned}$$

So

$$\begin{aligned} d(p_0, q_0) &\leq \sum_{j=0}^{n-1} d(p_j, p_{j+1}) \leq \sum_{j=0}^{n-1} (U(|x_j - x_{j+1}|) - (t_j + t_{j+1})) \\ &\leq 20\kappa s = 80\kappa s \end{aligned}$$

□

Proof. of Lemma 3.1.2 The claim is clear if $c_\alpha = 1$. Otherwise we know

$$\frac{c_\alpha}{c_\beta} = \frac{\left| \frac{b}{B} - \frac{a+b}{A+B} \right|}{\left| \frac{a}{A} - \frac{a+b}{A+B} \right|}$$

$c_\alpha \geq c_\beta$ therefore gives us that

$$\left| \frac{a}{A} - \frac{a+b}{A+B} \right| \leq \left| \frac{b}{B} - \frac{a+b}{A+B} \right|$$

- Suppose $\frac{b}{B} < \frac{a}{A}$. Writing $b = c_1 a$, $B = c_2 A$, we have

$$\begin{aligned} 1 - \frac{1+c_1}{1+c_2} &< \frac{1+c_1}{1+c_2} - \frac{c_1}{c_2} \\ 1 + \frac{c_1}{c_2} &< 2 \left(\frac{1+c_1}{1+c_2} \right) \\ c_2(1+c_2) + c_1(1+c_2) &< 2(1+c_1)c_2 \\ c_2 + c_2^2 + c_1 + c_1c_2 &\leq 2c_2 + 2c_1c_2 \\ c_2(c_2 - 1) &< c_1(c_2 - 1) \end{aligned}$$

So if $A < B = c_2 A$, then $1 < c_2$, and this gives us $c_2 < c_1$, which means $1 < \frac{c_1}{c_2}$. Multiplying both sides by $\frac{a}{A}$ this means $\frac{a}{A} < \frac{b}{B}$, contradiction. So $A \geq B$.

- now suppose $\frac{a}{A} < \frac{b}{B}$. Then again, that $\frac{a}{A}$ is closer to $\frac{a+b}{A+B}$ than $\frac{b}{B}$ means

$$\begin{aligned} \frac{a+b}{A+B} - \frac{a}{A} &< \frac{b}{B} - \frac{a+b}{A+B} \\ \frac{1+c_1}{1+c_2} - 1 &< \frac{c_1}{c_2} - \frac{1+c_1}{1+c_2} \\ 2 \left(\frac{1+c_1}{1+c_2} \right) &< 1 + \frac{c_1}{c_2} \\ 2c_2(1+c_1) &< c_2(1+c_2) + c_1(1+c_2) \\ 2c_2 + 2c_1c_2 &< c_2 + c_2^2 + c_1 + c_1c_2 \\ c_1(c_2 - 1) &< c_2(c_2 - 1) \end{aligned}$$

If $A < B$, then $c_2 > 1$, and this gives us $c_1 < c_2$, which means $\frac{c_1}{c_2} < 1$. Multiplying by $\frac{a}{A}$ this says $\frac{b}{B} < \frac{a}{A}$, contradiction. So $A \geq B$. □

Lemma 4.3.1. *Given a triangle in \mathbb{R}^2 with vertices A, B, C , and opposites of length a, b, c , satisfying $\frac{a+b}{c} \leq 1 + \epsilon$ for some $\epsilon \in [0, 0.5]$, then*

- $d(C, \overline{AB}) \leq 1.5\epsilon^{1/4}\overline{AB}$

- $\min\{A, B\} \leq \max\{\pi - \cos^{-1}(-1 + \sqrt{\frac{\epsilon}{1+\epsilon}}), \sin^{-1}(\frac{\sqrt{\frac{\epsilon}{1+\epsilon}}}{2})\}$

Proof. the condition on the length means

$$1 \geq \frac{c^2}{(a+b)^2} = \frac{(a+b)^2 - 2ab(1 + \cos(C))}{(a+b)^2} \geq \frac{1}{1+\epsilon}$$

Write $\frac{1}{1+\epsilon} = 1 - \hat{\epsilon}$, (note that $\hat{\epsilon} = 1 - \frac{1}{1+\epsilon} \leq \epsilon$) for some small $\hat{\epsilon} > 0$, we have

$$0 \leq \frac{2ab}{(a+b)^2}(1 + \cos(c)) \leq \hat{\epsilon}$$

which means EITHER

- $(1 + \cos(C)) \leq \sqrt{\hat{\epsilon}}$.

In this case, $\cos(C) \leq -(1 - \sqrt{\hat{\epsilon}})$, so $\cos^{-1}(-1 + \sqrt{\hat{\epsilon}}) \leq C \leq \pi$, leaving $A, B < A + B \leq \pi - \cos^{-1}(-1 + \sqrt{\hat{\epsilon}})$ giving

$$d(C, \overline{AB}) = |\overline{AC}| \sin(A) \leq |\overline{AB}| \sin(\pi - \cos^{-1}(-1 + \sqrt{\hat{\epsilon}})) = |\overline{AB}| \sin(\cos^{-1}(-1 + \sqrt{\hat{\epsilon}}))$$

Hence

$$d(C, \overline{AB}) \leq |\overline{AB}| \sqrt{1 - (1 - \sqrt{\hat{\epsilon}})^2} \leq |\overline{AB}| \sqrt{(1 - 1 + \sqrt{\hat{\epsilon}})(1 + 1 - \sqrt{\hat{\epsilon}})} \leq |\overline{AB}| \sqrt{2\sqrt{\hat{\epsilon}}}$$

OR

- $\frac{2ab}{(a+b)^2} \leq \sqrt{\hat{\epsilon}}$. By Sine rule, this is the same thing as

$$\frac{2 \sin(A) \sin(B)}{(\sin(A) + \sin(B))^2} \leq \sqrt{\hat{\epsilon}}$$

Divide top and bottom by $\sin(B)$ (if $\sin(A) = \sin(B) = 0$ then we are done, so assume one of them is not zero) so

$$2 \sin(A) \leq 2 \frac{\sin(A)}{\sin(B)} \leq \frac{2 \frac{\sin(A)}{\sin(B)}}{\left(1 + \frac{\sin(A)}{\sin(B)}\right)^2} \leq \sqrt{\hat{\epsilon}}$$

yields $A \leq \sin^{-1}\left(\frac{\sqrt{\hat{\epsilon}}}{2}\right)$. Since $\epsilon \leq 0.5$, $\hat{\epsilon} = 1 - \frac{1}{1+\epsilon} \leq \frac{1}{3}$. So $\angle A \leq 16.78^\circ$. Since $C + B = \pi - A$, WLOG $C \geq B$, $C \geq \frac{\pi - A}{2} \geq 45^\circ$ so $\tan(C) \geq 1$. Therefore

$$\begin{aligned} \frac{|\overline{AC}|}{|\overline{AB}|} &= \frac{\sin(B)}{\sin(C)} = \frac{\sin(\pi - C - A)}{\sin(C)} = \frac{\sin(\pi - C) \cos(A)}{\sin(C)} - \frac{\sin(A) \cos(\pi - C)}{\sin(C)} \\ &= \cos(A) + \frac{\sin(A)}{\tan(C)} \leq \cos(A) + \sin(A) \leq 2 \end{aligned}$$

Hence

$$d(C, \overline{AB}) = \sin(A) |\overline{AC}| \leq \sin(A) 2 |\overline{AB}| \leq \frac{\sqrt{\hat{\epsilon}}}{2} 2 |\overline{AB}| = \sqrt{\hat{\epsilon}} |\overline{AB}|$$

□

Proof. of Lemma 4.1.3 The quadrilateral is the same as the loop below.

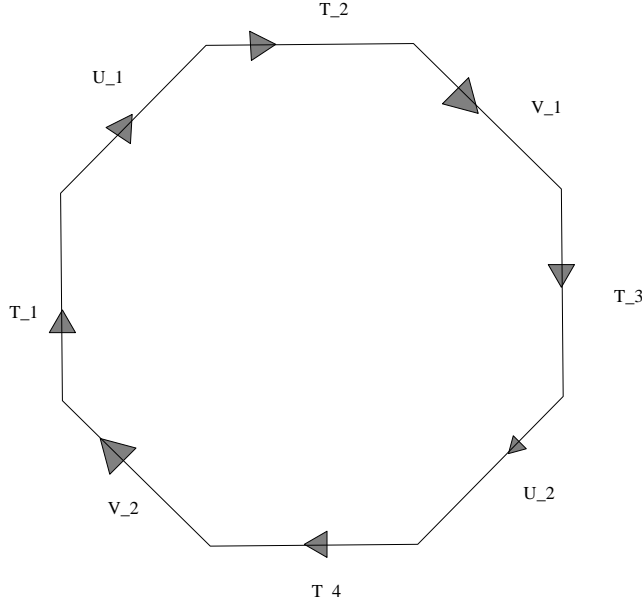


Figure 8: The loop given by a quadrilateral

Write $\mathbf{T}_i = T_i v$. Since $|U_1|, |U_2|, |V_1|, |V_2|$ are all less than $\eta(\sum |\mathbf{T}_i|)$, the first claim that $\sum_{i=1}^4 T_i \leq \eta(\sum_{i=1}^4 |\mathbf{T}_i|)$ follows by walking around the loop associated to Q .

So it cannot be the case that all the T_i 's are of the same sign. WLOG we can assume $T_2 > 0$, and $T_3 < 0$. Furthermore, regardless of the signs of the remaining T_i 's, there must be another pair of adjacent T_i 's of opposite signs, and either this pair involves one of $\{T_2, T_3\}$, or that it doesn't. In the latter case, $T_1 > 0$ and $T_4 < 0$, and the projection of this quadrilateral into $\langle v \rangle \times \mathbb{R}^m$ is a quadrilateral with two consecutive upward and two consecutive downward edges, and such a quadrilaterals doesn't exist.

So either T_2 or T_3 is involved in a pair of oppositely signed edges. WLOG, we assume $T_1 < 0$. Then by (iv) in the definition of a quadrilateral, we have that $d(e, \Pi_{W_v^+}(U_1)) \geq 1$, because $T_1 < 0$ and $T_2 > 0$; and $d(e, \Pi_{W_v^-}(V_1)) \geq 1$, because $T_2 > 0$ and $T_3 < 0$, where $\Pi_{W_v^+} : (x, t) \mapsto \pi_{W_v^+}(x)$, $\pi_{W_v^+}$ is the usual projection from \mathbb{R}^m to W_v^+ . $\Pi_{W_v^-}$ is defined similarly.

Suppose $T_4 < 0$. Then $|T_2| = |T_1| + |T_3| + |T_4|$. Writing the loop as:

$$\begin{aligned} e &= \mathbf{T}_2 V_1 \mathbf{T}_3 U_2 \mathbf{T}_4 V_2 \mathbf{T}_1 U_1 \\ &= (\mathbf{T}_2 V_1 \mathbf{T}_2^{-1})(\mathbf{T}_2 \mathbf{T}_3 U_2 \mathbf{T}_3^{-1} \mathbf{T}_2^{-1})(\mathbf{T}_2 \mathbf{T}_3 \mathbf{T}_4 V_2 \mathbf{T}_1) U_1 \end{aligned}$$

we see that only in the first bracket do we have a coordinate of size $e^{|T_2|}$. So $T_4 > 0$, and again by (iv) in the definition of quadrilateral, we conclude that for $i = 1, 2$, $d(e, \Pi_{W_v^+}(U_i)) \geq 1$, $d(e, \Pi_{W_v^-}(V_i)) \geq 1$. \square

Proof. of Lemma 4.1.4 Summing the \mathbb{R} coordinates we see that $r_0 + r_2 = r_1 + r_3$. The identity word can be written as

$$\begin{aligned} e &= (r_0, 0)u_0(-r_1, 0)u_1(r_2, 0)u_2(-r_3, 0)u_3 \\ &= ((r_0, 0)u_0(-r_0, 0))((r_0 - r_1, 0)u_1(r_1 - r_0, 0))((r_3, 0)u_2(-r_3, 0))u_3 \end{aligned}$$

we see that $|r_0 - r_3| \leq d(e, u_0) + d(e, u_2)$, and $|r_0 - r_1| \leq d(e, u_1) + d(e, u_3)$ by comparing the W^+ and W^- coordinates.

Similarly by looking at the word starting from $(-r_1, 0)$ we have

$$\begin{aligned} e &= (-r_1, 0)u_1(r_2, 0)u_2(-r_3, 0)u_3(r_0, 0)u_0 \\ &= ((-r_1, 0)u_1(r_1, 0))((-r_1 + r_2, 0)u_2(-r_2 + r_1, 0))((-r_0, 0)u_3(r_0, 0))u_0 \end{aligned}$$

which gives us that $|r_1 - r_0| \leq d(e, u_1) + d(e, u_3)$, and $|r_2 - r_1| \leq d(e, u_2) + d(e, u_0)$. We obtain the desired claim by writing the word starting at $(r_2, 0)$ and $(-r_3, 0)$ and argue similarly as above. \square

Proof. of Lemma 4.2.1 Equip the set $A \times B$ with the product measure $\mu = \mu_\alpha \times \mu_\beta$. The measure of the set $R = \{(a, b) : a \sim b\}$ is therefore $\mu(R) = \int_A \mu_\beta(B_a) d\mu_\alpha = \int_B \mu_\alpha(A_b) d\mu_\beta$. Hence

$$\frac{1}{M_B} \frac{\mu(R)}{\mu_\beta(B)} \leq \mu_\alpha(A_b)_{\min}, \quad \mu_\beta(B_a)_{\max} \leq \frac{\mu(R)}{\mu_\alpha(A)} M_A \quad (23)$$

Let χ be the characteristic function of the set $\{(a, b) : a \sim b, a \in A_s\}$. Then

$$\begin{aligned} \int_B \left(\int_{A_b} \chi d\mu_\alpha \right) d\mu_\beta &= \int_A \left(\int_{B_a} \chi d\mu_\beta \right) d\mu_\alpha = \int_{A_s} \mu_\beta(B_a) d\mu_\alpha \leq s \mu_\alpha(A) \mu_\beta(B_a)_{\max} \\ \int_B \left(\int_{A_b} \chi d\mu_\alpha \right) d\mu_\beta &\geq \int_{B^{s,t}} \left(\int_{A_b} \chi d\mu_\alpha \right) d\mu_\beta \geq t \int_{B^{s,t}} \mu_\alpha(A_b) d\mu_\beta \geq t \mu_\alpha(A_b)_{\min} \mu_\beta(B^{s,t}) \end{aligned}$$

Therefore

$$\mu_\beta(B^{s,t}) \leq \frac{s \mu_\alpha(A) \mu_\beta(B_a)_{\max}}{t \mu_\alpha(A_b)_{\min}} \leq \frac{s}{t} M_A M_B \mu_\beta(B)$$

where the last inequality comes from quoting equation(23) \square